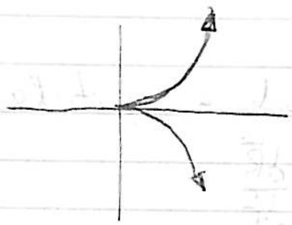
Getting rectangular (non-parametric) equations for a curve from parametric ones: eliminate t (if possible).

E.g., $y = t^3$, $x = t^2$ in the plane, t all $\mathbb{R}$.

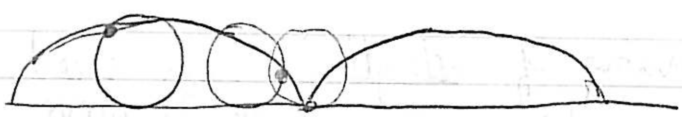
$x \geq 0$, $t = \pm\sqrt{x} \Rightarrow y = \pm x^{3/2}$ (two branches)



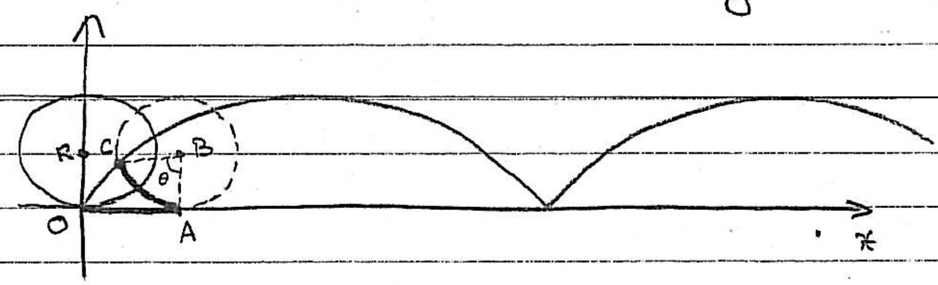
or implicitly as $y^2 - x^3 = 0$.

$$y = \frac{2t}{1+t^2}, \quad x = \frac{1-t^2}{1+t^2}$$

Cycloid



2. The cycloid is the curve described by a point on a wheel which is moving on the ground.



To find a parametrization of the cycloid we need a parameter. It can be the time, but it's easy to see that if the wheel has constant speed, the time is proportional with θ. We'll choose θ as parameter (the calculations are easier).

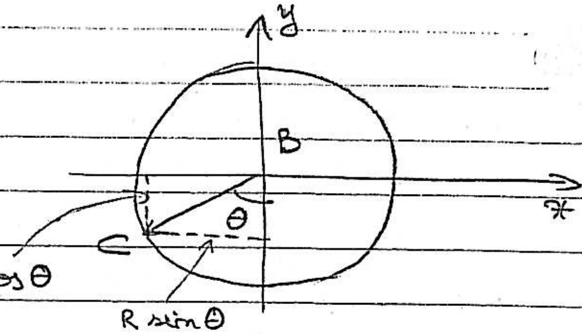
We describe the position vector of C.

$$\vec{OC} = \vec{OA} + \vec{AB} + \vec{BC}$$

$\vec{OA} = \text{arc } AC = R\theta \cdot \hat{i}$ (R - radius of wheel)

$\vec{AB} = R \cdot \hat{j}$

$$\overrightarrow{BC} = -R\sin\theta\,\hat{i} - R\cos\theta\,\hat{j}$$



(this is why vectors are good: to calculate $\overrightarrow{BC}$ we moved the system of coordinates to have the origin at B without hesitation; we get the same vector)

$$\overrightarrow{OC} = R(\theta - \sin\theta)\,\hat{i} + R(1-\cos\theta)\,\hat{j}$$

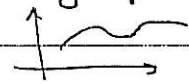
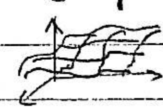
So:
$$\begin{cases} x = R(\theta - \sin\theta) \\ y = R(1-\cos\theta) \end{cases}$$

Let's find the length of the cycloid between

3. Functions of more variables

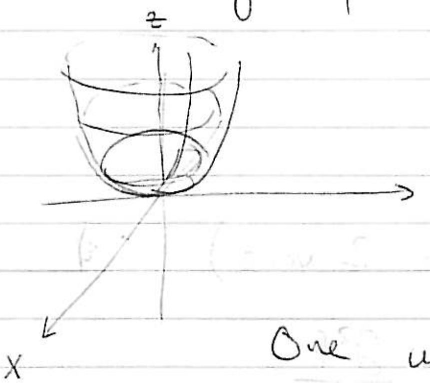
$f(x,y) = xy^2+2$, $f(x,y,z) = \sin x + \tan(yz)$ etc...

What do we want to study about functions of more variables. Precisely what we wanted for functions of one variables; there will be a few differences, but a lot of similarities.

1 variable	2 variables	3 variables
graph: a curve 	graph: a surface 	no graph
tangent line	tangent plane	—
slope of tangent	many slopes	
$f'(x)$ $f''(x), f'''(x)$	partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial f}{\partial z}, \frac{\partial^3 f}{\partial x \partial y \partial z}, \dots$	
max: $f'(x)=0$ $f''(x)<0$	max: " $f'(x)$ " $=0$ " $f''(x) < 0$ " (negative defined	
area under graph	volume under graph	—
length of arc	area of surface	—
$\int_a^b f(x)dx$	$\iint_R f(x,y)dxdy$	$\iiint_R f(x,y,z)dxdydz$

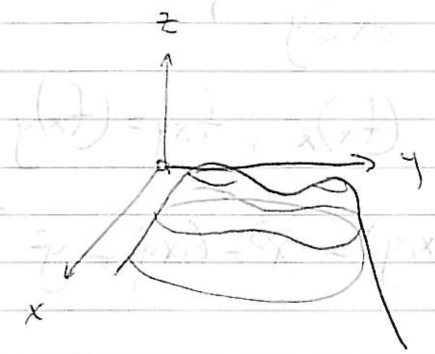
Graphing functions of 2 variables

Eg. $f(x,y) = x^2+y^2$ The graph $z=f(x,y)$ is the set of points $(x, y, f(x,y))$ in 3-space.

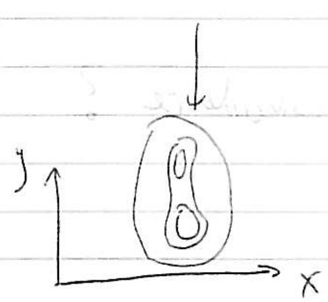


Hard to draw! One solution: use a computer. (E.g., Athena machines have Mathematica, Maple, MatLab, etc., I believe.)

One way to deal with this problem: contour plots



Take intersection of the graph w/ planes parallel to the xy-plane, i.e., w/ eqn $z =$ const. These give <u>contour lines</u>. Flattening them out gives the <u>level curves</u>, or the contour plot.



Sometimes, may want to label the lines to indicate their height.

Eg: what are the level curves for $f(x,y) = xy$?

For $g(x,y) = \sin x \sin y$?

(Think about what sort of slices we want to look at, i.e., how far apart.)

Equivalent idea:
Topographical maps.

Partial derivatives

Given $f(x,y)$, The partial derivatives are what we get by differentiating w.r.t. one variable while treating the others as constant.

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}\Big|_{(x,y)} = f_x(x,y)$$

Similarly $\frac{\partial f}{\partial y}$, ($\frac{\partial f}{\partial z}$ etc if more than 2 vars) &

higher derivatives (second partials) $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$,

f_{xx}, f_{xy}, etc. (Note $f_{xx} = (f_x)_x$, $f_{xy} = (f_x)_y$.)

E.g., what are the second partials of $f(x,y) = x^3 + 6xy + y^2$?

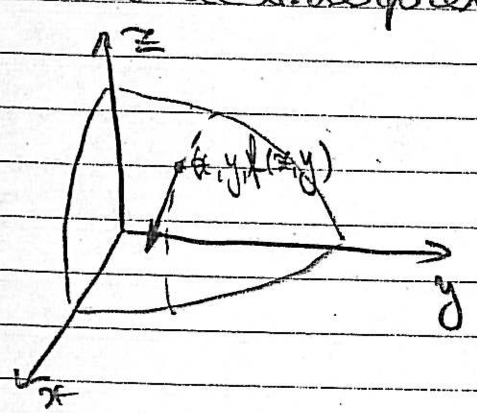
$$f_x = 3x^2 + 6y \longrightarrow f_{xx} = 6x, \quad f_{xy} = 6$$

$$f_y = 6x + 2y \longrightarrow f_{yx} = 6, \quad f_{yy} = 2$$

$f_{xy} = 6$, $f_{yx} = 6$ — Funny coincidence?

Thm: If f_{xy} & f_{yx} are both defined & continuous at a point, they are equal.

Geometric interpretation of $\frac{\partial f}{\partial x}$



Cut the graph with the plane $y = \text{const}$ and in that plane $\frac{\partial f}{\partial x}$ is the slope of the tangent to the curve.

3. The gradient

It is a useful idea to organize the first order partial derivatives in a vector, called the gradient

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

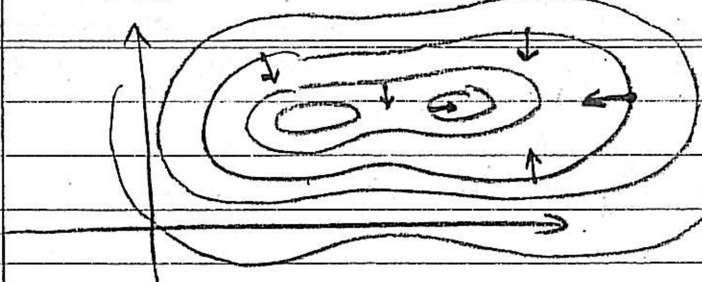
(if f is a function of 3 var,

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\Big)$$

Sometimes denoted by $\vec{\nabla} f|_{(x,y)}$ or $\vec{\nabla} f|_{(x,y,z)}$

Geometric interpretation.

Assume f has 2 variables. Draw the level curves



At any given point the gradient is perpendicular to the level curve passing through that point, and points towards where f increases; its magnitude show how fast f increases in that direction.

Take a point (x_0, y_0) and assume we have a parametrization of a level curve passing through that point, $x = x(t)$, $y = y(t)$. That means

$$f(x(t), y(t)) = \text{const}$$

$$\Rightarrow \frac{d}{dt} f(x(t), y(t)) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = 0$$

$$\Rightarrow \underbrace{\left(\frac{\partial f}{\partial x} \cdot \hat{i} + \frac{\partial f}{\partial y} \cdot \hat{j} \right)}_{\vec{\nabla} f} \underbrace{\left(\frac{\partial x}{\partial t} \cdot \hat{i} + \frac{\partial y}{\partial t} \hat{j} \right)}_{\vec{v}} = 0$$

$\Rightarrow \vec{\nabla} f \cdot \vec{v} = 0 \Rightarrow \vec{\nabla} f \perp \vec{v} \Rightarrow \vec{\nabla} f$ normal to the level curve

For a function of 3 variables, $\vec{\nabla} f$ will be normal to the <u>level surface</u>.

4. The chain rule.

Suppose $z = f(x,y)$ and $x = g(u,v)$
$y = h(u,v)$

$$\begin{tikzcd}
 & & u \\
 & x \arrow[ur] \arrow[r] & v \\
z \arrow[ur] \arrow[dr] & & \\
 & y \arrow[r] \arrow[dr] & u \\
 & & v
\end{tikzcd}$$

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}}$$

Exercises

a. $\frac{\partial z}{\partial u}$ if $z = e^{uv} \sin(u+v)$

b. $w = e^{x^2+y^2}$, $x = \cos t$, $y = \sin t$; $\frac{\partial w}{\partial t} = ?$

a. $w = x^2 + y^2$, $x = u^2 - v^2$, $y = 2uv$

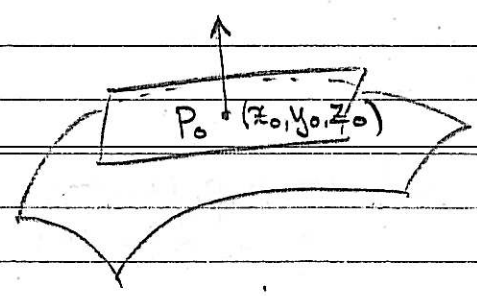
$$\frac{\partial w}{\partial u} = ?\ , \quad \frac{\partial w}{\partial v} = ?$$

4. The tangent plane

Suppose we have a surface given in the implicit form

$$F(x, y, z) = 0$$

Then $\vec{\nabla} F \big|_{(x_0, y_0, z_0)}$ is a vector normal to the surface at the point (x_0, y_0, z_0)

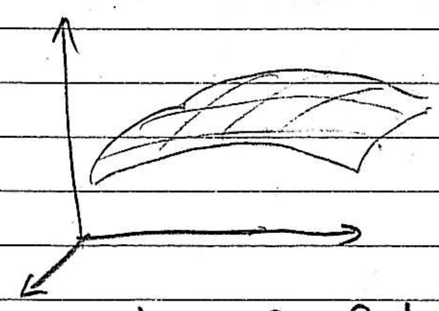


Then the equation of the tangent plane is:

$$F_x\big|_{P_0}(x - x_0) + F_y\big|_{P_0}(y - y_0) + F_z\big|_{P_0}(z - z_0) = 0$$

$$\left(\vec{\nabla} F\big|_{P_0} = F_x\big|_{P_0}\,\hat{i} + F_y\big|_{P_0}\,\hat{j} + F_z\big|_{P_0}\,\hat{k}\right)$$

If the surface is the graph of a function of 2 variables



$$z = f(x, y)$$

We calculate the equation of the tangent plane by reducing to the previous case.

$$F(x, y, z) = z - f(x, y)$$

$$\vec{\nabla} F = (-f_x, -f_y, 1)$$

$\Rightarrow$ equation of tangent plane:

$$-f_x\big|_{P_0}(x - x_0) - f_y\big|_{P_0}(y - y_0) + (z - z_0) = 0$$

$$\Rightarrow \boxed{z - z_0 = f_x\big|_{P_0}(x - x_0) + f_y\big|_{P_0}(y - y_0)}$$

Exercise: Find equation of tangent plane to the paraboloid $z = x^2 + y^2$ at $(1, 2, 5)$

5. Linear approximation

$$z = f(x, y)$$

Just as in the case of one variable, we use the equation of the tangent plane

$$z - z_0 = f_x\big|_{P_0}(x - x_0) + f_y\big|_{P_0}(y - y_0)$$

to approximate $f(x, y)$

$$\boxed{f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)\cdot(x - x_0) + f_y(x_0, y_0)\cdot(y - y_0)}$$

Exercise: Approximate linearly $\theta = \arctan \frac{y}{x}$ around $(3, 4)$. Is θ more sensitive to changes in x or in y?