Two other important ideas:

1) Using series in limits: $\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{x - (x - \frac{x^3}{6} + \cdots)}{x^3}$

$= \lim_{x \to 0} \frac{x^3/6 - \cdots}{x^3} = 1/6$. (Easier than L'Hop often!)

2) Taylor series can also be centered at any point $a$:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Exam 2 material is now finished.

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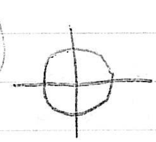
Hold off on this section — do lines & planes first.

Vector functions & parametric equations

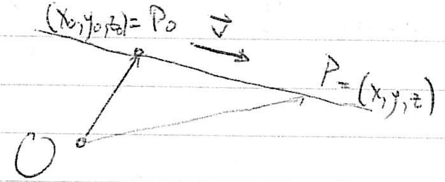
So far, we've talked a lot about functions $y = f(x)$ giving us curves in the x-y plane; but not every curve is associated to a function, e.g., (ask for examples) . ← given by 2 different fns, or implicit equation, or parametric equations

$y = \sin t$, $x = \cos t$ (so $x, y$ are both functions of a third variable, $t$.) Equivalently, a single vector-valued function

$F(t) = \langle \cos t, \sin t \rangle$.

Lines & planes in 3-space.

A line is determined by a point $P_0$ & a direction (nonzero vector parallel to it) $\vec{v}$. Every point on a line




can be found by starting at $P_0$ & moving in direction $\vec{v}$ for some distance

$\Rightarrow$ There is some $t$, $\overrightarrow{P_0P} = t\vec{v} = t\cdot\langle a,b,c\rangle$

$$\langle x-x_0, y-y_0, z-z_0\rangle = \langle ta, tb, tc\rangle$$

So every point on the line is given by

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

We can also eliminate $t$ between these (parametric) equations to get $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$, the symmetric equations for the line.

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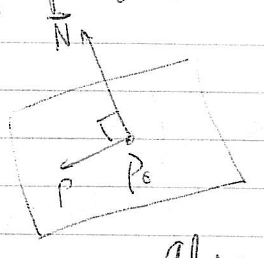
A plane is determined by a point $P_0$ & a <u>normal vector</u> $\vec{N}$ perpendicular to it. For any $P=(x,y,z)$ in plane,




$\overrightarrow{P_0P} \perp \vec{N} \Rightarrow \vec{N}\cdot\overrightarrow{P_0P} = 0$

$\Rightarrow a(x-x_0)+b(y-y_0)+c(z-z_0)=0$,

equiv $ax+by+cz=d$

Also, a plane is determined by 3 noncolinear points, $P_0, P_1, P_2$. Get $\vec{N}$ how?

$$\boxed{\vec{N} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}}.$$

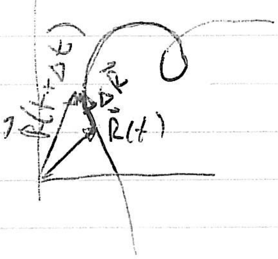
Exercises: • Line through $(1,-1,0)$ & $(2,1,-2)$? Does this line contain $(3,3,4)$?

Plane through $(1,0,0)$, $(2,1,0)$ & $(-1,0,1)$? Intersection of this plane & line?

Now go back & do parametric.

We can also do the same thing to describe motion of a particle in 3-space. Its position $(x(t), y(t), z(t))$ will be a function of $t$ (time), & the position vector is

$$\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$$




The <u>velocity</u> (vector) of such a particle is

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{R}}{\Delta t} =: \frac{d\vec{R}}{dt}$$

Coordinatewise, this is precisely what you would hope for:

$$\vec{v} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle x', y', z' \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle = \frac{d\vec{R}}{dt}.$$

(dot = Newton's notation = physicists notation = derivative w.r.t. time).

The <u>speed</u> is just the length of the velocity vector,

$$|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$= \sqrt{\left(\frac{ds}{dt}\right)^2}$$

$$= \frac{ds}{dt}$$ = derivative of <u>arclength</u>, i.e., speed really gets exactly how fast you're moving along the curve.

<u>Note</u>: velocity is always tangent to the curve (Newton's rule that a particle with no force acting on it would continue in a straight line)

so $\hat{v} = \hat{T} = \dfrac{\vec{v}}{|\vec{v}|}$ is the <u>unit tangent</u>, &

$$\vec{v} = \frac{ds}{dt} \cdot \vec{T}.$$

Acceleration $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{R}}{dt^2} = \left\langle \dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}, \dfrac{d^2z}{dt^2} \right\rangle = \langle \ddot{x}, \ddot{y}, \ddot{z} \rangle.$