18.089 Instructor: Joel Lewis

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$t \rightsquigarrow f(t)$

$x \rightsquigarrow y(x)$

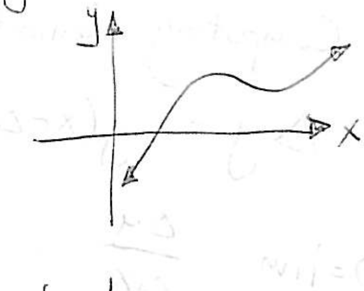
$s, t \rightsquigarrow f(s,t)$

input(s) $\rightsquigarrow$ function value

Summarize by a graph

f a function

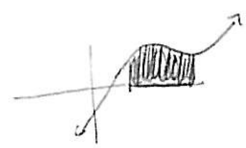
$y = f(x)$



Calculus: 2 geometric problems, 1 new tool

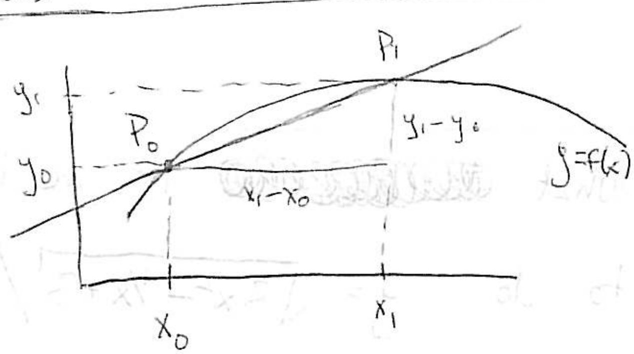
Problem 1: Given a curve, find its tangent lines (differential calculus)

Problem 2: Given a region bounded by curves, find its area (integral calculus)



Tool: The idea of a limit: $\lim_{x \to a} f(x)$ = "The value that $f(x)$ approaches as x gets close to a, if it exists."

Examples



$\overleftrightarrow{P_0P_1}$ is a secant line with slope $\dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$.

As $x_1 \to x_0$, the secant line approaches the tangent line $\Rightarrow$

$f'(x_0) = $ slope of tangent line $= \lim_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

definition

"The derivative of f", "f prime", $\dfrac{df}{dx}$, $\left.\dfrac{df}{dx}\right|_{x=x_0}$

Other interpretations: avg. vs. instantaneous rate of change; velocity

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Computing derivatives from the definition.

$\Delta y = y(x+\Delta x) - y(x)$

$y'(x) = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$, & we try to algebraically simplify $\frac{\Delta y}{\Delta x}$ to compute this derivative.

Examples: $y(x) = x^2$ $\qquad$ $y(x) = 3$

Binomial theorem & $y(x) = x^n$ $\qquad$ $y(x) = \sqrt{x}$

$y(x) = \frac{1}{x}$

What ~~[illegible]~~ if we had to do $y = \sqrt{3x^2 - 1/x + 3}\,/(2x+7)$?

Need additional tools.

First, can generalize above results to get $\frac{d}{dx} x^n = n \cdot x^{n-1}$.

Second, if c is a constant and $f(x) = c \cdot g(x)$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c\,g(x+\Delta x) - c\,g(x)}{\Delta x} = c \cdot g'(x).$$

Third, $\frac{d}{dx}\left(g(x) + h(x)\right) = g'(x) + h'(x)$, (assuming both exist).

Now we have three more complicated rules.

- Product rule. If f, g are differentiable functions then

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Proof: $\Delta(fg) = (f + \Delta f)(g + \Delta g) - fg$

$= \Delta f \cdot g + f \cdot \Delta g + \Delta f \cdot \Delta g$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta(fg)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \cdot g + f \cdot \frac{\Delta g}{\Delta x} + \frac{\Delta f}{\Delta x} \cdot \Delta g$$

$= f' \cdot g + f \cdot g' + f' \cdot 0$

$= f' \cdot g + f \cdot g'$.

Exercise: compute $(x^2 - 4)(x+1)$ in two different ways.

- Chain rule: If $y = f(u)$ & $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$ or equivalently if $y = f(g(x))$

then $y' = f'(g(x)) \cdot g'(x)$.

[skip proof, in text].

Examples: Compute $\frac{d}{dx}\left((x^2+4)^4\right)$ in two different ways.

Compute $\frac{d}{dx}\left(\sqrt{x^2+1}\right)$

- Quotient rule If $y(x) = \frac{f(x)}{g(x)}$ then

$$y'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Proof: $\frac{f(x)}{g(x)} = f(x)\cdot(g(x))^{-1}$, so

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \overset{P.R.}{=} f'(x)\cdot(g(x))^{-1} + f(x)\cdot\frac{d}{dx}\left(g(x)^{-1}\right)$$

$$\overset{C.R.}{=} f'(x)/g(x) + f(x)\cdot(-1)(g(x))^{-2}\cdot g'(x)$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

∎

Examples: Compute $\frac{d}{dx}\left(\frac{1}{x^3}\right)$

Compute $\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$

Compute $\frac{d}{dx}\left((x+1)^{100}\right)$.

Applications of derivatives / Higher derivatives.

If $y = f(t)$ is the position, $\frac{dy}{dt} = f'(t)$ is instantaneous rate of change, i.e., velocity. Then

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(f'(t)\right)$$

measures how fast velocity is changing, i.e., acceleration.

$$\frac{d^2y}{dt^2} = f''(t)$$

Similarly, could consider higher derivatives: $\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right) = \frac{d^3y}{dt^3} = f'''(t)$

$$\frac{d^ny}{dt^n} = f^{(n)}(t).$$

Examples: Compute y'' & y''' of

- $y = \sqrt{x}$
- $y = x^2$
- $y = x^3$
- $y = f(g(x))$

Graphing: Graph $y = f(x)$.

1st derivative
- $f'(x) > 0 \Rightarrow$ increasing
- $f'(x) < 0 \Rightarrow$ decreasing
- $f'(x) = 0$ or $f'(x)$ DNE $\Rightarrow$ critical point
  - max
  - min
  - inflection point
  - cusp
  - discontinuity

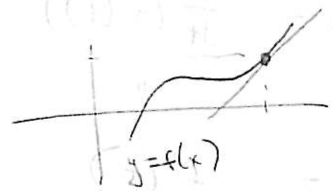
2nd derivative
- $f''(x) > 0 \Rightarrow$ concave up
- $f''(x) < 0 \Rightarrow$ concave down
- $f''(x) = 0 \Rightarrow$ inflection point (change in concavity)

Especially, note 2nd derivative test: if $f'(x) = 0$ & $f''(x) < 0$, local max

if $f'(x) = 0$ & $f''(x) > 0$, local min

Example: graph $y = x^3 - 3x + 2$ with maxima, minima, inflection points.

Tangent lines (The geometric problem we started with):



What is the tangent line to the curve $y = \sqrt{x^2+9}$ at the point $(4,5)$?

Idea: slope is given by derivative. $y' = \frac{x}{\sqrt{x^2+9}}$,

so $y'(4) = \frac{4}{5}$. $\Rightarrow$ line is $y - 5 = \frac{4}{5}(x-4)$ or $y = \frac{4}{5}x + \frac{9}{5}$.

In general: $y = f(x)$ @ (x_0, y_0) (where $y_0 = f(x_0)$),

slope is $f'(x_0)$ NB: not x! so equation is

$$y - y_0 = f'(x_0) \cdot (x - x_0) \quad \text{or} \quad y = f(x_0) + f'(x_0) \cdot (x - x_0).$$

Tangent line = linear approximation to the function near x_0.

Implicit differentiation: Suppose $x^2 + y^2 = 4$ (so y not given as a fn. of x). What are the slopes of the tangent lines at points where $x=1$?

Idea: instead of solving for x, can differentiate the given equation:

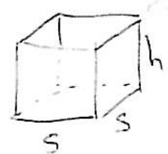
$$x^2 + y^2 = 4 \quad \text{so} \quad 2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Suppose instead $-y^3 + y + x^2 + x = 0$. What is the tangent line at the point $(2,2)$?

Some basic sorts of problems we can solve using this new tool are

- Max-min problems: have some function f & we want to find its extrema (maxima or minima):

Suppose we want to build an open rectangular box w/ a square base & volume = 250 cc. What's the smallest amount of material necessary (i.e., least surface area) to do this?



$f(s,h) = s^2 + 4sh$ ← to minimize under the condition

$500 = V = s^2 h$. First, eliminate variables to have a single-variable function to minimize: $h = \frac{500}{s^2}$

$\Rightarrow$ minimize $s^2 + \frac{2000}{s} = f(s)$.

Minimum $\Rightarrow$ either boundary value (not in this case) or $f'(s) = 0$
$0, \infty$

Solve
$$f'(s) = 2s - \frac{2000}{s^2} = 0 \Rightarrow s^3 = 1000 \Rightarrow s = 10 \quad (\& \text{ so } h = 5).$$

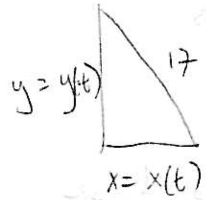
Check this is min w/ 2nd derivative test: $f''(s) = 2 + \frac{4000}{s^3}$.

$f''(10) = 6 > 0$ ✓

- Related rates problems: Two changing quantities (functions of time) are related (by some equation). Know how fast one is changing, want to know how fast other is changing.

Ladder of length 17 ft leans against a wall. Its feet start to slide away from the wall at 0.3 ft/sec. When the top of the ladder is 15 ft above the ground, how fast is it falling?

Strategy: draw a picture, identify quantities in question.



Given $\frac{dx}{dt} = \frac{3}{10}$. How are x & y related?

$$x^2 + y^2 = 17^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

When $y = 15$, $x = 8$, $\frac{dx}{dt} = \frac{3}{10} \Rightarrow \frac{dy}{dt} = -\frac{x \cdot \frac{dx}{dt}}{y} = -\frac{8 \cdot 3/10}{15}$

$$= -4/25 \text{ ft/sec.}$$

Q: why the $-$ sign?

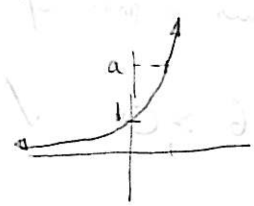
Lecture 1 ended here ~~(3 hrs)~~ (3 hrs)

Some special numbers and functions

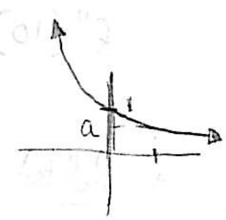
Exponential functions: $f: \mathbb{R} \to (0, \infty)$ given by $f(x) = a^x$ where a is a positive real number. (called the base of the exponential).

Three types of behavior:

- If $a = 1$, $f(x) = 1$ for all x $\to$ constant function. We will just ignore this one from now on.
- If $a > 1$, an increasing function $\to$



- If $a < 1$, a decreasing function $\to$



The inverse function (check this idea!) of the exponential function with base a is the Logarithm with the base a, $g: (0, \infty) \to \mathbb{R}$

$g(x) = \log_a x$. (Here $a > 0$, $a \neq 1$.)

$$\text{So } a^{\log_a x} = \log_a(a^x) = x.$$