18.089 Exam 3

75 minutes

Tuesday, June 30, 2009

Name:

This exam consists of six problems, some of which have many parts, not arranged in any particular order. (Note that pages are printed front and back!) Please solve all problems in the space provided, showing all work as neatly and clearly as possible. **Be sure to read the problems carefully!**

You are allowed two (8.5 by 11 inch or equivalent) pages of notes for this exam. You may use a calculator, but please no sophisticated graphing or algebraic manipulation capabilities. All work must be your own.

Problem	Value	Score
Problem 1	15	
Problem 2	10	
Problem 3	22	
Problem 4	28	
Problem 5	15	
Problem 6	10	
Total	100	

Problem 1. *(15 points)* Consider the function $f(x, y) = \dfrac{\sin y}{x^2+1}$. This function has a unique critical point (x_0, y_0) with the property that $-1 \le x_0 \le 1$ and $0 \le y_0 \le 3$. Find this critical point and determine whether it is a maximum, minimum or saddle point.

$$f_x = \frac{-2x \sin y}{(x^2+1)^2} = 0$$

$$f_y = \frac{\cos y}{x^2+1} = 0$$

$$\frac{\cos y}{x^2+1} = 0 \Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2} + n\pi \text{ for some}$$

integer n $\Rightarrow y = \pi/2$.

Then $f_x = \dfrac{-2x}{(x^2+1)^2} \sin \pi/2 = 0$

$\Leftrightarrow x = 0$, so

$$\boxed{(x_0, y_0) = (0, \pi/2).}$$

$$f_{xx} = \frac{-2\sin y\,(x^2+1)^2 - (-2x\sin y)\cdot 4x(x^2+1)}{(x^2+1)^4}$$

$$f_{xy} = \frac{-2x\cos y}{(x^2+1)^2}$$

@ $(0, \pi/2)$, $f_{xx} = -2$

$f_{xy} = 0$

$$f_{yy} = \frac{-\sin y}{x^2+1}$$

$f_{yy} = -1$

$\Rightarrow H = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ $\det H = 2 > 0$,

$f_{xx} < 0 \Rightarrow$ $\boxed{\text{maximum}}$.

Problem 2. *(10 points)* Write down a system of equations whose solutions give the points at which the function $f(x, y, z) = x^2 + 4yz$ reaches its extrema, subject to the constraint $x^2 + y^2 + z^2 = 1$. (You do not need to solve this system.)

$$f_x = 2x \qquad g_x = 2x$$
$$f_y = 4z \qquad g_y = 2y$$
$$f_z = 4y \qquad g_z = 2z$$

$$\Rightarrow \quad \boxed{\begin{aligned} 2x &= 2x \cdot \lambda \\ 4z &= 2y \cdot \lambda \\ 4y &= 2z \cdot \lambda \\ x^2+y^2+z^2 &= 1 \end{aligned}}$$

(To solve: from $2^{nd}, 3^{rd}$ eqns get $4y = \lambda^2 y \Rightarrow$ $y = 0$ or $\lambda = 2$ or $\lambda = -2$. From 1^{st} eqn, $x = 0$ or $\lambda = 1$.

$$y = 0 \overset{2^{nd}}{\Downarrow} z = 0 \qquad \lambda = 2 \overset{2^{nd}}{\Downarrow} y = z \qquad \lambda = -2 \overset{2^{nd}}{\Downarrow} y = -z$$

	$y=z=0$	$y=z$, $\lambda=2$	$y=-z$, $\lambda=-2$
$x=0$	$\times$	$\pm(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$\pm(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
$\lambda=1$	$(1,0,0)$ $(-1,0,0)$	$\times$	$\times$

$$f(1,0,0) = f(-1,0,0) = 1$$

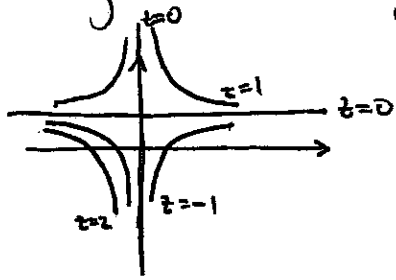
$$f(0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = f(0, -\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}) = 2 \leftarrow \text{max}$$

$$f(0, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) = f(0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}) = -2 \leftarrow \text{min}$$

Problem 3. *(4 + 4 + 4 + 4 + 2 + 4 = 22 points)* All parts of this problem refer to the surface with equation $z = xy - x$.

(a) Sketch four level curves of the surface, at height (z-value) -1, 0, 1 and 2.

$z = k \Rightarrow xy - x = k \Rightarrow y = 1 + \frac{k}{x}$ (unless $x = 0$, which only happens when $k = 0$)



(b) Find and classify (maximum, minimum, saddle point) all critical points of the surface.

$z_x = y - 1 = 0$

$z_y = x = 0$

$\Rightarrow (x,y) = (0,1)$

$z_{xx} = 0$

$z_{yy} = 0$

$z_{xy} = 1$

$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\det H = -1 \Rightarrow$ unique crit pt is saddle pt @ $(0,1)$.

(c) Compute the gradient of the surface at the point $(1, 1, 0)$.

$$@ \ (1,1,0), \quad \vec{\nabla} z = \langle y-1, x \rangle \Big|_{(1,1)} = \boxed{\langle 0, 1 \rangle}$$

(d) Compute the tangent plane to the surface at the point $(1, 1, 0)$.

$$z - 0 = 0 \cdot (x - 1) + 1 \cdot (y - 1)$$

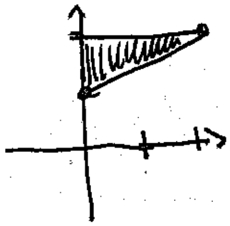
$$\rightarrow \boxed{z = y - 1}$$

(e) Compute the directional derivative of the surface at the point $(1, 1, 0)$ in the direction $\langle -2, 3 \rangle$.

$$\hat{u} = \frac{\langle -2, 3 \rangle}{|\langle -2, 3 \rangle|} = \frac{\langle -2, 3 \rangle}{\sqrt{13}}$$

$$\vec{\nabla} z \cdot \hat{u} = \langle 0, 1 \rangle \cdot \frac{\langle -2, 3 \rangle}{\sqrt{13}} = \frac{3}{\sqrt{13}}$$

(f) Compute the volume of the solid bounded below the surface and above the triangular region R bounded by the lines $y = 2$, $x = 0$ and $y = \frac{x}{2} + 1$.



$$\int_0^2 \int_{\frac{x}{2}+1}^{2} xy - x \, dy \, dx$$

$$= \int_0^2 \frac{xy^2}{2} - xy \Big|_{\frac{x}{2}+1}^{2} dx$$

$$= \int_0^2 (2x - 2x) - \left(\frac{x}{2}\left(\frac{x}{2}+1\right)^2 - x\left(\frac{x}{2}+1\right)\right) dx$$

$$= \int_0^2 \frac{x}{2} - \frac{x^3}{8} \, dx$$

$$= \frac{x^2}{4} - \frac{x^4}{32} \Big|_0^2$$

$$= \left(1 - \frac{1}{2}\right) - 0$$

$$= \frac{1}{2}.$$

Or $$\int_1^2 \int_0^{2y-2} xy - x \, dx \, dy = \int_1^2 \frac{x^2}{2}(y-1) \Big|_0^{2y-2} dy = \int_1^2 \frac{(2y-2)^2}{2}(y-1) dy$$

$$= \int_1^2 2(y-1)^3 dy = \frac{1}{2}(y-1)^4 \Big|_1^2 = \frac{1}{2}.$$

Problem 4. *(4 + 4 + 4 + 4 + 4 + 4 + 4 = 28 points)* All parts of this problem refer to the curve given by the parametric equations $x = t^2$, $y = t^3$, $z = t^5 - t$. Note that this curve passes through the point $(1,1,0)$ when $t = 1$.

(a) Find the velocity of the curve at the point $(1,1,0)$.

$\vec{v}(t) = \langle 2t, 3t^2, 5t^4 - 1 \rangle$

@ $t = 1$, $\boxed{\vec{v} = \langle 2, 3, 4 \rangle}$

(b) Find the acceleration of the curve at the point $(1,1,0)$.

$\vec{a}(t) = \langle 2, 6t, 20t^3 \rangle$

$\boxed{\vec{a}(1) = \langle 2, 6, 20 \rangle}$

(c) Find the speed of the curve when at the point $(1,1,0)$.

speed $= |\vec{v}| = \sqrt{2^2 + 3^2 + 4^2} = \boxed{\sqrt{29}}$

(d) Find the unit tangent vector to the curve at the point $(1, 1, 0)$.

$$\boxed{\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{29}} \langle 2, 3, 4 \rangle}$$

(e) Find the symmetric equations for the tangent line to the curve at the point $(1, 1, 0)$.

The line has direction vector $\langle 2,3,4 \rangle$ & passes through $(1,1,0)$ so it has equations

$$\frac{x-1}{2} = \frac{y-1}{3} = \frac{z}{4}$$

(f) The curve also passes through the point $(0,0,0)$. Write down an integral that gives the arclength of the curve between $(0,0,0)$ and $(1,1,0)$.

The curve passes through $(0,0,0)$ when $t=0$, so the arclength is given by

$$S = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^1 \sqrt{(2t)^2 + (3t^2)^2 + (5t^4-1)^2}\, dt$$

(g) Find all intersection points of the curve with the surface $z = xy - x$ of Problem 4.

$z = xy - x$ & $\langle x, y, z\rangle = \langle t^2, t^3, t^5 - t\rangle$

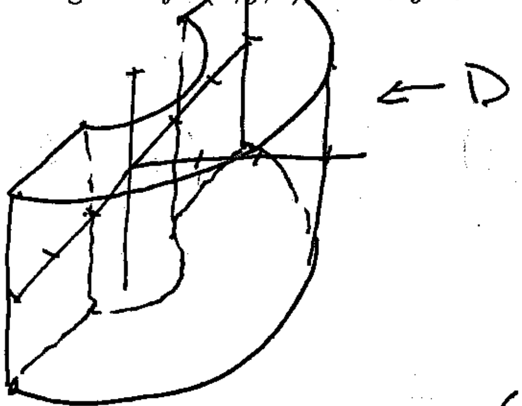
$\Rightarrow \quad t^5 - t = t^2 \cdot t^3 - t^2$

$\Rightarrow \quad t^2 = t$

$\Rightarrow \quad t = 1$ or 0.

So the surface & curve intersect at $(0,0,0)$ & $(1,1,0)$.

Problem 5. *(15 points)* Let D be the solid consisting of points (x, y, z) that satisfy $y \geq 0$, $1 \leq x^2 + y^2 \leq 9$ and $-1 \leq z \leq 1$. (This solid looks like half of a cylindrical shell.) The density of D is given by $\delta(x, y, z) = x^2 + y^2 + z^2$. Compute the y-coordinate of the center of mass of D.



Cylindrical coordinates!

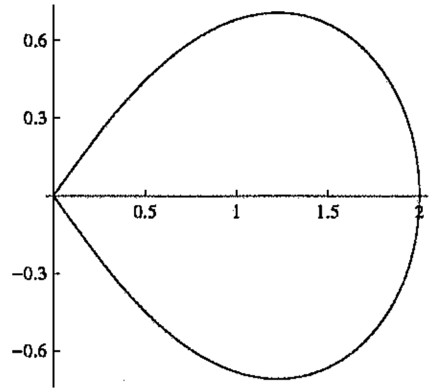
$1 \leq r \leq 3 \qquad 0 \leq \theta \leq \pi$

$-1 \leq z \leq 1$

$$\bar{y} = \frac{\iiint_D y \cdot \delta \, dV}{\iiint_D \delta \cdot dV}$$

$$= \frac{\int_0^{\pi}\int_1^3\int_{-1}^1 r\sin\theta \cdot (z^2+r^2)\, r \, dz\, dr\, d\theta}{\int_0^{\pi}\int_1^3\int_{-1}^1 (z^2+r^2)\, r\, dz\, dr\, d\theta}$$

Problem 6. *(10 points)* A curve given by the polar equation $r = 2\sqrt{\cos 2\theta}$ for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ is shown below. Compute the area it bounds.



We have $$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} 2\cos 2\theta \, d\theta$$

$$= \sin 2\theta \Big|_{-\pi/4}^{\pi/4}$$

$$= 1 - (-1)$$

$$= \boxed{2}.$$