18.089 Exam 2

75 minutes

Wednesday, June 17, 2009

Name: Solutions

This exam consists of eight problems, not arranged in any particular order. (Note that pages are printed front and back, including the back of the last page!). Please solve all problems in the space provided, showing all work as neatly and clearly as possible. **Be sure to read the problems carefully!**

You are allowed one (8.5 by 11 inch or equivalent) page of notes for this exam. You may use a calculator, but please no sophisticated graphing or algebraic manipulation capabilities. All work must be your own.

Problem	**Value**	**Score**
Problem 1	11	
Problem 2	11	
Problem 3	15	
Problem 4	10	
Problem 5	10	
Problem 6	15	
Problem 7	12	
Problem 8	16	
Total	100	

Problem 1. *(5 + 6 = 11 points)* Let $M = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ -b & 0 & a \end{pmatrix}$.

(a) Compute $M \cdot M$.

Solution. Using our standard matrix multiplication algorithm, we find

$$\boxed{M \cdot M = \begin{pmatrix} a^2 - b^2 & 0 & 2ab \\ 0 & 1 & 0 \\ -2ab & 0 & a^2 - b^2 \end{pmatrix}}.$$

□

(b) Compute M^{-1}.

Solution. The matrix of minors of M is

$$\begin{pmatrix} a & 0 & b \\ 0 & a^2 + b^2 & 0 \\ -b & 0 & a \end{pmatrix}.$$

Multiplying by the checkerboard changes the signs only of the zero entries, so doesn't change anything. Taking the transpose gives

$$\begin{pmatrix} a & 0 & -b \\ 0 & a^2 + b^2 & 0 \\ b & 0 & a \end{pmatrix}.$$

Finally, we have $\det M = a^2 + b^2$, so

$$\boxed{M^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & 0 & -b \\ 0 & a^2 + b^2 & 0 \\ b & 0 & a \end{pmatrix}}.$$

□

Problem 2. *(5 + 6 = 11 points)* Let $\mathbf{U} = \langle 1, 1, 0\rangle$, $\mathbf{V} = \langle 1, 0, 1\rangle$.
(a) Compute the angle between $\mathbf{U}$ and $\mathbf{V}$.

Solution. If θ is the angle between $\mathbf{U}$ and $\mathbf{V}$ then we have that $\cos\theta = \frac{\mathbf{U}\cdot\mathbf{V}}{|\mathbf{U}|\cdot|\mathbf{V}|}$. We have $|\mathbf{U}| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$ and $|\mathbf{V}| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$ and $\mathbf{U}\cdot\mathbf{V} = 1 + 0 + 0 = 1$, so $\cos\theta = \frac{1}{2}$ and $\boxed{\theta = \dfrac{\pi}{3}}$. □

(b) Compute $\mathbf{W} = \mathbf{U} \times \mathbf{V}$ and then compute $\mathbf{U}\cdot\mathbf{W}$.

Solution. We have

$$\begin{aligned}\mathbf{W} &= \mathbf{U} \times \mathbf{V} \\ &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \boxed{\mathbf{i} - \mathbf{j} - \mathbf{k}}.\end{aligned}$$

Thus $\mathbf{U}\cdot\mathbf{W} = 1 - 1 + 0 = \boxed{0}$. We also know that $\mathbf{U}\cdot\mathbf{W}$ must be 0 because as the cross product of $\mathbf{U}$ with some other vector, $\mathbf{W}$ must be perpendicular to $\mathbf{U}$. □

Problem 3. *(10 + 5 = 15 points)* Define $I_n = \displaystyle\int x(\ln x)^n\,dx$.

(a) Use integration by parts to write I_n in terms of I_{n-1}.

Solution. Set $u = (\ln x)^n$ and $dv = x\,dx$. Then $du = n(\ln x)^{n-1}\frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$. Thus

$$\begin{aligned}
\int x(\ln x)^n\,dx &= \int u\,dv \\
&= uv - \int v\,du \\
&= \frac{x^2(\ln x)^n}{2} - \int \frac{x^2}{2}\cdot n(\ln x)^{n-1}\frac{1}{x}\,dx \\
&= \frac{x^2(\ln x)^n}{2} - \frac{n}{2}\int x(\ln x)^{n-1}\,dx \\
&= \frac{x^2(\ln x)^n}{2} - \frac{n}{2}\cdot I_{n-1},
\end{aligned}$$

so

$$\boxed{I_n = \frac{x^2(\ln x)^n}{2} - \frac{n}{2}\cdot I_{n-1}}.$$

$\square$

(b) Compute I_3. (It might be helpful to use $I_0 = \int x\,dx$.)

Solution. Using part (a) repeatedly, we have

$$\begin{aligned}
I_3 &= \frac{x^2(\ln x)^3}{2} - \frac{3}{2}I_2 \\
&= \frac{x^2(\ln x)^3}{2} - \frac{3}{2}\left(\frac{x^2(\ln x)^2}{2} - \frac{2}{2}I_1\right) \\
&= \frac{x^2(\ln x)^3}{2} - \frac{3}{2}\left(\frac{x^2(\ln x)^2}{2} - \frac{2}{2}\left(\frac{x^2(\ln x)^1}{2} - \frac{1}{2}I_0\right)\right) \\
&= \frac{x^2(\ln x)^3}{2} - \frac{3}{2}\left(\frac{x^2(\ln x)^2}{2} - \frac{2}{2}\left(\frac{x^2(\ln x)^1}{2} - \frac{1}{2}\cdot\frac{x^2}{2}\right)\right) \\
&= \boxed{\frac{x^2(\ln x)^3}{2} - \frac{3x^2(\ln x)^2}{4} + \frac{3x^2\ln x}{4} - \frac{3x^2}{8}}.
\end{aligned}$$

$\square$

Problem 4. *(10 points)* Compute the limit $\lim_{x\to 0^+} x(\ln x)^2$. (Note that this problem has nothing whatsoever to do with problem 3.)

Solution. The limit is a $0\cdot\infty$ indeterminant form. We rewrite it as

$$\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{x^{-1}},$$

a $\frac{\infty}{\infty}$ indeterminant form, and now we apply L'Hopital's rule to get

$$\begin{aligned}\lim_{x\to 0^+} \frac{(\ln x)^2}{x^{-1}} &= \lim_{x\to 0^+} \frac{2\frac{\ln x}{x}}{-x^{-2}} \\ &= \lim_{x\to 0^+} -2x\ln x.\end{aligned}$$

We now apply the same trick again to get

$$\begin{aligned}\lim_{x\to 0^+} -2x\ln x &= \lim_{x\to 0^+} -2\frac{\ln x}{x^{-1}} \\ &= \lim_{x\to 0^+} -2\frac{\frac{1}{x}}{-x^{-2}} \\ &= \lim_{x\to 0^+} 2x \\ &= \boxed{0}.\end{aligned}$$

□

Problem 5. *(10 points)* Does the integral

$$\int_1^\infty \frac{(x^2+1)\ln x}{x^3}\,dx$$

converge or diverge? (Explain!)

Solution. We have that for all $x > e$, $\ln x > 1$ and so

$$\frac{(x^2+1)\ln x}{x^3} > \frac{x^2}{x^3} = \frac{1}{x}.$$

Since the integral $\int_1^\infty \frac{1}{x}\,dx$ diverges by the p-test ($p = 1 \le 1$), we have that the given integral $\boxed{\text{diverges by comparison}}$. □

Problem 6. *(5 + 5 + 5 = 15 points)* Decide whether the following series converge or diverge. (Explain!)

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(n^2+1)\ln n}{n^3}$.

Solution. The function $f(n) = \frac{(n^2+1)\ln n}{n^3}$ is positive and decreasing for $n \geq 2$, so the integral test applies. Then we have that this series $\boxed{\text{diverges by the integral test and the previous problem}}$. ☐

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(n^2+1)\ln n}{n^3}$.

Solution. The function $f(n) = \frac{(n^2+1)\ln n}{n^3}$ is positive and decreasing for $n \geq 2$, and its limit is 0, so we have that this sum $\boxed{\text{converges by the alternating series test}}$. ☐

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{3n}{n+1} \cdot \frac{1}{2^n}$.

Solution. We may show that this series converges either by the limit comparison test (comparing with $\frac{1}{2^n}$) or by the ratio test:

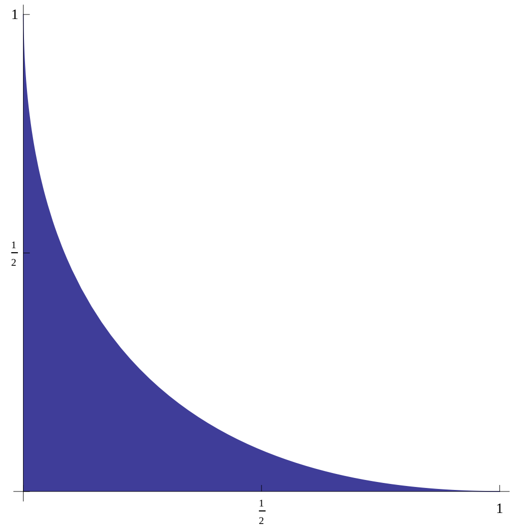
$$\begin{aligned}\lim_{n\to\infty} \left| \frac{\frac{3(n+1)}{n+2} \cdot \frac{1}{2^{n+1}}}{\frac{3n}{n+1} \cdot \frac{1}{2^n}} \right| &= \lim_{n\to\infty} \frac{(n+1)^2}{2n(n+2)} \\ &= \frac{1}{2} \\ &< 1,\end{aligned}$$

so the sum $\boxed{\text{converges by the ratio test}}$. ☐

Problem 7. *(12 points)* Consider the region in the first quadrant bounded by the curve

$$y = (1 - \sqrt{x})^2$$

for $0 \leq x \leq 1$, shown below. This region is rotated around the x-axis, forming a solid of rotation. Write down **two integrals** that give the volume of this solid, one using disks and the other using shells. *You do not need to evaluate these integrals!*



Solution. Using disks, we have that the radius of the disk is $y = (1 - \sqrt{x})^2$ and its thickness is dx. Thus the volume of an individual disk is

$$\pi \left((1 - \sqrt{x})^2\right)^2 \, dx$$

and the total volume is

$$\boxed{\int_0^1 \pi \left((1 - \sqrt{x})^2\right)^2 \, dx}.$$

Using shells, we have that the radius of the shell is y, its height is $x = (1 - \sqrt{y})^2$ (which we get by solving the equation defining the curve for x) and its thickness is dy, so its volume is

$$2\pi y (1 - \sqrt{y})^2 \, dy$$

and the total volume is

$$\boxed{\int_0^1 2\pi y (1 - \sqrt{y})^2 \, dy}.$$

Both of these integrals evaluate to $\dfrac{\pi}{15}$.

□

Problem 8. *(16 points)* Compute the first four nonzero terms (that is, the constant, x, x^2 and x^3 terms) of the power series for the function

$$f(x) = e^{x+x^2}$$

using any method you like.

First solution. We compute derivatives to get that

$$\begin{aligned} f'(x) &= (2x+1)e^{x+x^2}, \\ f''(x) &= (4x^2+4x+3)e^{x+x^2} \end{aligned}$$

and

$$f'''(x) = (8x^3+12x^2+18x+7)e^{x+x^2}.$$

(One needn't multiply out the polynomials, of course.) Thus $f(0) = 1, f'(0) = 1, f''(0) = 3$ and $f'''(0) = 7$, so

$$\boxed{f(x) = 1 + x + \frac{3x^2}{2} + \frac{7x^3}{6} + \dots}.$$

$\square$

Second solution. An exponent rule gives $f(x) = e^x \cdot e^{x^2}$. We have $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$ and so $e^{x^2} = 1 + x^2 + \dots$. Thus

$$\begin{aligned} f(x) &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right) \cdot \left(1 + x^2 + \dots\right) \\ &= \boxed{1 + x + \frac{3x^2}{2} + \frac{7x^3}{6} + \dots}. \end{aligned}$$

$\square$

Third solution. We use composition of functions to get that

$$\begin{aligned} f(x) &= 1 + (x+x^2) + \frac{(x+x^2)^2}{2} + \frac{(x+x^2)^3}{6} + \dots \\ &= 1 + (x+x^2) + \frac{x^2 + 2x^3 + \dots}{2} + \frac{x^3 + \dots}{6} + \dots \\ &= \boxed{1 + x + \frac{3x^2}{2} + \frac{7x^3}{6} + \dots}. \end{aligned}$$

$\square$