18.089 Exam 1

75 minutes

Friday, June 5, 2009

Name: Solutions

This exam consists of seven problems, not arranged in any particular order. (Note that pages are printed front and back). Please solve all problems in the space provided, showing all work as neatly and clearly as possible.

This exam is open-note and open-text. You may use a calculator so long as it lacks sophisticated graphing capabilities. All work must be your own.

Problem	**Value**	**Score**
Problem 1	10	
Problem 2	15	
Problem 3	20	
Problem 4	10	
Problem 5	10	
Problem 6	15	
Problem 7	20	
Total	100	

Problem 1. *(10 points)* Compute the derivative $\frac{d}{dx}\ln\left(3^x+1\right)$.

Solution. We have $\frac{d}{du}\ln u = \frac{1}{u}$, so by the chain rule

$$\frac{d}{dx}\ln\left(3^x+1\right) = \frac{\frac{d}{dx}\left(3^x+1\right)}{3^x+1}.$$

Now recall that we computed $\frac{d}{dx}\left(a^x\right) = a^x\ln(a)$, or just recall the method:

$$\begin{aligned}\frac{d}{dx}\left(a^x\right) &= \frac{d}{dx}\left(e^{x\ln a}\right)\\ &= e^{x\ln a}\cdot\ln a\\ &= a^x\ln a.\end{aligned}$$

Thus the answer is $\dfrac{3^x\ln 3}{3^x+1}$. □

Problem 2. *(15 points)* Compute the arclength of the curve $y = \ln\cos x$ for $\frac{\pi}{6}\le x\le\frac{\pi}{4}$.

Solution. We apply the known formula

$$s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

with $\frac{dy}{dx} = -\tan x$, $a = \frac{\pi}{6}$ and $b = \frac{\pi}{4}$. This gives

$$\begin{aligned}s &= \int_{\pi/6}^{\pi/4}\sqrt{1+(-\tan x)^2}\,dx\\ &= \int_{\pi/6}^{\pi/4}\sqrt{1+\tan^2 x}\,dx\\ &= \int_{\pi/6}^{\pi/4}\sqrt{\sec^2 x}\,dx\\ &= \int_{\pi/6}^{\pi/4}\sec x\,dx\\ &= \ln(\sec x+\tan x)\Big|_{x=\pi/6}^{\pi/4}\\ &= \ln(\sqrt{2}+1)-\ln(\sqrt{3}).\end{aligned}$$

□

Problem 3. *(20 points)* Compute $\displaystyle\int_3^4 \frac{x^3+4x^2}{(x-1)(x^2-4x+8)}\,dx$.

Solution. Since the degree of the numerator is the same as the denominator, we first need to carry out long division. Dividing x^3+4x by $(x-1)(x^2-4x+8) = x^3-5x^2+12x-8$ yields a quotient of 1 and a remainder of $9x^2-12x+8$, so we may rewrite our integral in question as

$$\int_3^4 1 + \frac{9x^2-12x+8}{(x-1)(x^2-4x+8)}\,dx.$$

We now apply partial fractions to the second summand, noting that x^2-4x+8 does not have real roots and so does not factor: we have

$$\frac{9x^2-12x+8}{(x-1)(x^2-4x+8)} = \frac{A}{x-1} + \frac{Bx+C}{x^2-4x+8},$$

and multiplying out gives

$$9x^2-12x+8 = A(x^2-4x+8) + (Bx+C)(x-1).$$

Setting $x=1$ gives $5=5A$ and so $A=1$. Substituting this value and expanding gives

$$9x^2-12x+8 = (B+1)x^2 + (C-B-4)x + (8-C)$$

and so (equating coefficients) $B=8$, $C=0$. This reduces our original integral to

$$\begin{aligned}\int_3^4 1 + \frac{1}{x-1} + \frac{8x}{x^2-4x+8}\,dx &= \int_3^4 1\,dx + \int_3^4 \frac{1}{x-1}\,dx + \int_3^4 \frac{8x}{x^2-4x+8}\,dx \\ &= 1 + (\ln 3 - \ln 2) + \int_3^4 \frac{8x}{x^2-4x+8}\,dx \\ &= 1 + \ln\frac{3}{2} + \int_3^4 \frac{8x}{x^2-4x+8}\,dx.\end{aligned}$$

We complete the square in the denominator to find $x^2-4x+8 = (x-2)^2+4$, and this suggests the substitution $x-2 = 2\tan u$. This means $dx = 2\sec^2 u\,du$, $x=3$ when $u = \arctan\frac{1}{2}$, and $x=4$ when $u = \arctan 1 = \frac{\pi}{4}$. Thus our expression becomes

$$\begin{aligned}1 + \ln\frac{3}{2} + \int_3^4 \frac{8x}{x^2-4x+8}\,dx &= 1 + \ln\frac{3}{2} + 8\int_{\arctan\frac{1}{2}}^{\frac{\pi}{4}} \frac{2+2\tan u}{4\tan^2 u + 4}\cdot 2\sec^2 u\,du \\ &= 1 + \ln\frac{3}{2} + 8\int_{\arctan\frac{1}{2}}^{\frac{\pi}{4}} 1 + \tan u\,du \\ &= 1 + \ln\frac{3}{2} + 8\left(u + \ln\cos u\Big|_{\arctan\frac{1}{2}}^{\frac{\pi}{4}}\right) \\ &= 1 + \ln\frac{3}{2} + 8\left(\left(\frac{\pi}{4} + \ln\frac{\sqrt{2}}{2}\right) - \left(\arctan\frac{1}{2} + \ln\cos\arctan\frac{1}{2}\right)\right) \\ &= 1 + 2\pi - 8\arctan\frac{1}{2} + \ln\frac{6144}{625}.\end{aligned}$$

(In the final step we used $\cos\arctan\frac{1}{2} = \frac{2}{\sqrt{1^2+2^2}}$, which follows from the Pythagorean Theorem.) ☐

Problem 4. *(10 points)* What is the tangent line to the curve $y = 2 - e^{2-2x}$ at $x = 1$?

Solution. The line passes through the point $(1, 2 - e^{2-2x}) = (1, 1)$ and has slope

$$\begin{aligned} y'(1) &= 2e^{2-2x}\Big|_{x=1} \\ &= 2, \end{aligned}$$

so it has equation $y - 1 = 2(x - 1)$ or $y = 2x - 1$. □

Problem 5. *(10 points)* What is the third derivative of $f(x) = \sin x \cos x$?

Solution. There are many ways to go about this problem. The most straightforward is to never use a double-angle formula but to repeatedly use the product rule or the chain rule and power rule to get

$$\begin{aligned} f'(x) &= \cos x \cdot \cos x - \sin x \cdot \sin x = \cos^2 x - \sin^2 x, \\ f''(x) &= 2\cos x(-\sin x) - 2\sin x \cos x = -4\cos x \sin x, \\ f'''(x) &= -4(-\sin x)\sin x + (-4)\cos x \cdot \cos x = 4\sin^2 x - 4\cos^2 x. \end{aligned}$$

Somewhat slicker is to rewrite $\sin x \cos x = \frac{1}{2}\sin 2x$ and then use the chain rule three times to get

$$\begin{aligned} f'(x) &= \cos 2x, \\ f''(x) &= -2\sin 2x, \\ f'''(x) &= -4\cos 2x. \end{aligned}$$

□

Problem 6. *(15 points)* A special balloon has the property that no matter its volume, it always maintains a perfectly cubical shape. The balloon is being inflated at a rate of 6 cubic centimeters per second. When the balloon has volume 64 cubic centimeters, how fast is its surface area increasing?

Solution. Let s denote the side-length of the cube, V the volume and SA the surface area, all considered as functions of time. The intended solution (which almost no one followed) was to write $V = s^3$ and $SA = 6s^2 = 6V^{\frac{2}{3}}$, so

$$\frac{dSA}{dt} = 6 \cdot \frac{2}{3} V^{-\frac{1}{3}} \cdot \frac{dV}{dt}.$$

At the moment in question we have $V = 64$ and $\frac{dV}{dt} = 6$, so

$$\frac{dSA}{dt} = 6 \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot 6 = 6.$$

It's also not substantially more difficult to compute $\frac{ds}{dt}$ as an intermediate step:

$$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$$

and so

$$\frac{ds}{dt} = \frac{6}{3s^2} = \frac{2}{s^2}.$$

Thus

$$\begin{aligned} \frac{dSA}{dt} &= 12s \frac{ds}{dt} \\ &= 12s \cdot \frac{2}{s^2} \\ &= \frac{24}{s} \\ &= 6. \end{aligned}$$

□

Problem 7. *(20 points)* Compute the values of a and b so that the function f defined by

$$f(x) = \begin{cases} x^3, & x \geq 1 \\ x^2 + ax + b, & x < 1 \end{cases}$$

is differentiable for all values of x. For this value of a and b, sketch the curve defined by $y = f'(x)$.

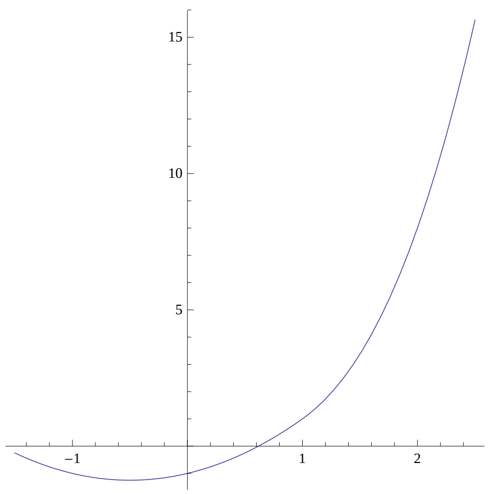
Solution. In order for the function to be differentiable it must certainly be continuous. The only place where this might be a problem is at $x = 1$, where the function value is 1 but

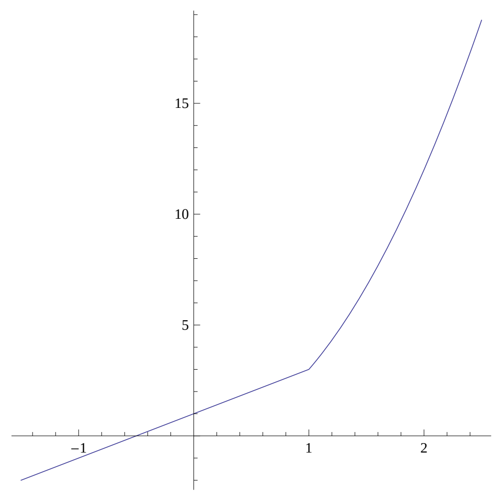
$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x^2 + ax + b \\ &= 1 + a + b. \end{aligned}$$

Thus we must have $1 + a + b = 1$ and so $a + b = 0$. The function is also clearly differentiable everywhere except possibly at $x = 1$, with derivative

$$f'(x) = \begin{cases} 3x^2, & x > 1 \\ 2x + a, & x < 1. \end{cases}$$

The derivative will exist at the point $x = 1$ if and only if the limit of the derivative from the left and right are equal, so we must also have $3 = 2 + a$. Thus $a = 1$ and $b = -1$. Below are graphs of $f(x)$ (note the near-seamlessness of the curve at $x = 1$) and $f'(x)$.





□