18.089 Homework Problems, Part 3

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88. Let $a > 0$ be some constant and let C be the circle of radius a centered at the origin. Let

$$\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right).$$

Compute $\oint_C \vec{F} \cdot d\vec{R}$ by choosing a parametrization and integrating. Is this vector field conservative?

89. We know that if the vector field $\vec{F}$ is conservative then there exists some scalar field f such that $\vec{F} = \vec{\nabla} f$. This means that if $\vec{F} = \langle F_1, F_2 \rangle$ then $F_1 = \frac{\partial f}{\partial x}$ and $F_2 = \frac{\partial f}{\partial y}$, and so $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. This means that if these partials are *not* equal, then $\vec{F}$ is *not* conservative. Use this method to show that the following fields are not conservative:

   (a) $\vec{F} = y\mathbf{i} - x\mathbf{j}$

   (b) $\vec{F} = \langle x^3 y, xy^2 \rangle$

90. Shows that $\vec{F} = \langle 2xy, x^2 \rangle$ is conservative by finding a scalar field f with $\vec{\nabla} f = \vec{F}$.

91. Use Green's Theorem to compute the following line integrals:

   (a) $\oint_C (xy - y^2)\, dx + xy^2\, dy$ where C is the boundary of the triangle with vertices $(0,0)$, $(1,1)$ and $(1,0)$.

   (b) $\oint_C x\, dx + xy^2\, dy$ where C is the simple closed path formed by the curves $y = x$ and $y = x^2$.

92. Use the formula we gave near the end of class on Thursday to write down an integral that gives the area bounded by the curve with parametric equations $x = \cos^3 t$, $y = \sin^3 t$ for $0 \le t \le 2\pi$. (You don't actually have to compute the integral.)

93. Use Green's Theorem to find the simple closed curve C (with positive orientation) for which

$$\oint_C (yx^2 + 3x)\, dx + (2xy - xy^2)\, dy$$

has the maximum value. (Hint: once you convert to a double integral, note that the integrand takes positive values in some places and negative values in other places. In order to make the integral as large as possible, what region should C enclose?)

94. Let $\vec{F} = \langle 2xy(1-x^2), y^2(3x^2-1) \rangle$. For each of the following curves, find the flux of $\vec{F}$ across the curve.

(a) The line segment joining $(-1, 1)$ to $(1, 1)$.

(b) The lower half of the unit circle $y = -\sqrt{1-x^2}$ joining $(-1, 0)$ to $(1, 0)$.

95. Show that the vector field $\vec{F}(x, y) = \langle ye^{xy} + 2, xe^{xy} - 1\rangle$ is conservative, and find a scalar field $f(x, y)$ such that $\vec{\nabla} f = \vec{F}$.

96. Which of the following three-dimensional vector fields are conservative? (You *don't* need to compute the associated scalar fields.)

(a) $\langle y + 2xz, x + z^2, x^2 + 2yz\rangle$

(b) $\langle 1 + y + yz, x + xz, xy + y\rangle$

97. Compute the divergence and curl of the following vector fields.

(a) $\vec{F} = \langle 2x^2y, 3xz^3, xy^2z^2\rangle$

(b) $\vec{G} = e^{x+y}\,\hat{\imath} + e^z \cos y\,\hat{\jmath} + e^z \sin y\,\hat{k}$

(c) $\vec{H} = \langle 3, x + y + z, x^2 + y^2 + z^2\rangle$

98. Use the Divergence Theorem to find the flux of the given vector field over the given surface S.

(a) $\vec{F} = \langle x, y, z\rangle$ and S is the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(b) $\vec{F} = \langle xz, y^2, x\rangle$ and S is the surface of the tetrahedron whose vertices are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 2)$. (It may be helpful to note that the faces of this tetrahedron are the coordinate planes together with the plane $2x + 2y - z = 0$.)

(c) $\vec{F} = \langle x, -y, z\rangle$, and S is the cylinder whose curved surface has equation $x^2 + y^2 = r^2$ and whose bases have equations $z = 0$ and $z = b$.

99. Suppose that S is a closed surface in a region of space where the vector field $\vec{F}$ is defined. Explain why

$$\iint_S (\operatorname{curl} \vec{F}) \cdot \hat{n}\, dA = 0.$$

100. Compute each of the following integrals by applying Stokes' Theorem to a well-chosen surface whose boundary is the given curve. (Assume the given curves are oriented counter-clockwise when looking down.)

(a) $\vec{F} = y(x-z)\hat{\imath} + (2x^2+z^2)\hat{\jmath} + y^3 \cos(xz)\hat{k}$ where C is the boundary of the square $0 \le x \le 2$, $0 \le y \le 2$, $z = 5$.

(b) $\vec{F} = \langle y - x, x - z, x - y\rangle$ and C is the boundary of the triangular part of the plane $x + y + 2z = 2$ that lies in the first octant. (Hint: there are two possible surfaces here; one is the triangle itself, while the other consists of triangular portions of three of the coordinate planes.)

101. Let S be the top half of the ellipsoid $x^2 + y^2 + \frac{z^2}{9} = 1$, oriented so that $\hat{n}$ is directed upward. If $\vec{F} = \langle x^3, y^4, z^3 \sin xy\rangle$, evaluate

$$\iint_S (\operatorname{curl} \vec{F}) \cdot \hat{n}\, dA$$

by replacing S with a simpler surface. (Don't forget to mention why the surface you choose is acceptable ... something about its boundary.)