

18.089 Homework Problems, Part 2

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50. Sketch the plane curve given by the parametric equations $y = \sin t$, $x = \sqrt{t}$ for $t \geq 0$.
51. A particle has position vector at time t given by $\mathbf{R}(t) = t^2\mathbf{i} + t^3\mathbf{j}$. Compute its velocity, acceleration and speed as functions of t .
52. A particle has position vector at time t given by $\mathbf{R}(t) = \langle \cos t, \sin t, t \rangle$. Compute its velocity, acceleration and speed as functions of t .
53. Give the parametric equation of the line of intersection of the planes $x - 2y + z = 2$ and $2x + y + 3z = 3$.
54. What is the intersection point of the line $\frac{x-1}{1} = \frac{y}{2} = \frac{z+2}{-2}$ and the plane $x + y + z = 3$?
55. An ellipse centered at the origin has rectangular equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Give some parametric equations for the same curve.
56. Draw contour plot with a few level curves (three or four well-chosen heights should suffice) for the following functions:
- (a) $z = 2x - y$
 - (b) $z = x^2 - y^2$
 - (c) $z = x^3 - y$
57. A function f is said to satisfy *Laplace's equation* if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

For the following functions, verify by direct calculation that they satisfy $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and that they satisfy Laplace's equation.

- (a) $f(x, y) = e^{-3x} \cos 3y$
 - (b) $f(x, y) = \arctan \frac{y}{x}$
 - (c) $f(x, y) = e^x \sin y$
58. Find the equation of the tangent plane to the following surfaces at the indicated points.
- (a) $z = \arctan \frac{y}{x}$ at $(4, 4, \pi/4)$.

(b) $z = e^x \cos y$ at $(0, 0, 1)$.

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ at (x_0, y_0, z_0) . (This surface is called an *ellipsoid*.)

59. Recall that the element of arclength of a curve parametrized by $\vec{R}(t) = \langle x(t), y(t) \rangle$ is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Compute the arclength of a single arch of the cycloid. (While integrating, you may need to use one of the half-angle identities – see, for example, http://en.wikipedia.org/wiki/List_of_trigonometric_identities .)

60. Let $w = \frac{y}{x}$, where $y = r \sin \theta$ and $x = r \cos \theta$ for parameters r, θ . Compute $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ in two ways: once by chain rule and once by direct substitution.

61. What are the directional derivatives of the following functions in the given directions at the given points?

(a) $f(x, y) = xy - x - y$ at $(1, 1)$ in the direction $\langle -1, 1 \rangle$.

(b) $f(x, y) = \frac{\arctan x}{\pi + \arctan y}$ at $(-1, 1)$ in the direction $\langle 2, 1 \rangle$.

(c) $f(x, y, z) = x^2y + y^2z + z^2x$ at $(1, -2, 2)$ in the direction $\langle 1, 1, 1 \rangle$.

62. Find all critical points of the following functions, and classify them as maxima, minima or saddle points or “impossible to tell from 2nd derivative test.” (It may help to remember that $a \cdot b = 0$ implies that either $a = 0$ or $b = 0$ or both.)

(a) $f(x, y) = (x^3 - x)(y^3 + 3y)$

(b) $f(x, y) = x^3 + y^3 + 3xy + 5$

(c) $f(x, y) = x^4 + y^4$

One of these functions has an “impossible to tell from 2nd derivative test” critical point. Is this point actually a maximum, minimum or saddle point?

63. It is sometimes convenient to define functions by integrals of the form

$$F(x) = \int_a^b f(x, y) dy.$$

If f is “nice,” we can calculate the derivative $F'(x)$ by “differentiating under the integral sign”:

$$F'(x) = \frac{d}{dx} \int_a^b f(x, y) dy = \int_a^b \left(\frac{\partial}{\partial x} f(x, y) \right) dy.$$

In parts (a) and (b), verify that this works by computing the integral, then the derivative directly and comparing to what you get by differentiating under the integral sign. In part (c), just compute the derivative $F'(x)$ using this method.

(a) $F(x) = \int_0^1 x + y dy$

(b) $F(x) = \int_0^\pi \sin xy dy$

$$(c) F(x) = \int_0^1 \frac{e^{x(t+1)}}{t+1} dt$$

64. If $z = \sin x + \sin 2y + \sin 3(x + y)$, is z more sensitive to changes in x or in y at $(0, 0)$?
65. If the equations $f(x) = 0$ and $f'(x) = 0$ have no common roots, show that every critical point of the function $z(x, y) = yf(x) + e^x$ is a saddle point.
66. If the sum of the three numbers x , y and z is equal to 12, what is the largest possible value for the product $x \cdot y^2 \cdot z^3$?
67. What is the equation of the plane passing through $(1, 2, 3)$ that cuts a tetrahedron from the first octant of maximal volume?
68. Find the point on the plane $x + 2y + 3z = 6$ that is closest to the origin. (Hint to make the arithmetic nicer: the distance is $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, and since $d \geq 0$, it is minimized at the same points where d^2 is minimized.)
69. Find the maximum and minimum values of the function $f(x, y) = x^2 - xy + y^2$ takes on the circle $x^2 + y^2 = 1$.
70. Find the values of a and b such that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has minimal area among all ellipses that pass through the point $(4, 1)$.

71. Consider the iterated integral $\int_0^1 \int_0^{\sqrt{x}} f(x, y) dy dx$. Draw the region in the plane that is being integrated over. Write the integral that arises from reversing the order of integration.

72. For each of the following iterated integrals, sketch the region being integrated over and compute the integral.

$$(a) \int_0^1 \int_{x^2}^x (2x + 2y) dy dx$$

$$(b) \int_1^{e^3} \int_0^{\frac{1}{y}} e^{xy} dx dy$$

$$(c) \int_1^2 \int_x^{2x} \frac{dy dx}{(x + y)^2}$$

73. Change the order of integration in the following iterated integrals. (You don't need to evaluate.)

$$(a) \int_0^2 \int_0^{4-x^2} 2xy dy dx$$

$$(b) \int_1^{e^3} \int_{\ln y}^3 2 dx dy$$

74. Calculate the value $\iint_R x dA$ when R is the first-quadrant part of the ring between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, $0 < a < b$. (Use either order of integration.)

75. Set up a double integral whose value is the volume of the solid under the function $z = \sqrt{1 - y^2}$ and above the disk $x^2 + y^2 \leq 1$. Choose an order of integration and integrate. (Hint: one way may be easier than the other.)
76. The curve given in polar coordinates by $r = 1 - \sin \theta$ is called a *cardioid*. Sketch it and compute the area that it bounds.
77. The curve given in polar coordinates by $r = 1 + 2 \sin \theta$ is called a *limaçon*. Sketch it – note in particular the existence of an inner and outer loop. Compute the area bounded by the inner loop.
78. Compute the following double integral by converting it to polar coordinates:

$$\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} x^2 + y^2 dy dx.$$

Here a is some positive constant.

79. Compute $\int_0^1 \int_0^{x^2} \int_0^{xy^3} 18x^3y^2z dz dy dx$.

80. Find the appropriate bounds to rewrite the triple integral

$$\int_0^a \int_0^x \int_0^y f(x, y, z) dz dy dx$$

with the order of integration $dx dy dz$.

81. Use an integral in cylindrical coordinates to find the mass of the solid bounded above the surface $z = 1 - x^2 - y^2$ and below the xy -plane whose density is given by $\delta = c(r^2 + z^2)$ for some positive constant c .
82. Complete the problem that we began near the end of class: if R is the solid that you get by drilling a radius-1 hole through a radius-2 hemisphere, what is its centroid? (Don't forget to compute the volume!)
83. Figure out what solid is given in spherical coordinates by the equation $\rho = 2 \sin \varphi$.
84. Assume that a, b and α are constants such that $0 < a < b$ and $0 < \alpha < \pi$. Compute the volume of the region bounded by the concentric spheres $\rho = a$ and $\rho = b$ and by the cone $\varphi = \alpha$.
85. (a) Find the mass of a solid sphere of radius a given that the density at a point is equal to the distance from that point to the surface of the sphere.
 (b) Would the answer to the preceding problem be larger or smaller if we replaced the word "surface" with "center"?
86. Evaluate the line integral $\int_C xy^2 dx - (x + y) dy$ where C is
 (a) the straight line segment from $(0, 0)$ to $(1, 2)$;
 (b) the parabolic path $y = 2x^2$ from $(0, 0)$ to $(1, 2)$;

(c) the broken line from $(0, 0)$ to $(1, 0)$ to $(1, 2)$.

Sketch all three paths.

87. Let $\mathbf{F} = \langle 2xy, x^2 + y^2 \rangle$. Compute $\int_C \mathbf{F} \cdot d\vec{P}$ where C is the semicircular path given by $x = \cos t$, $y = \sin t$ for $0 \leq t \leq \pi$.