18.089 Homework Problems

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1. We say that $\lim_{x \to a^+} f(x)$ is the limit of $f(x)$ as x approaches a from the right (or from above), and similarly $\lim_{x \to a^-} f(x)$ is the limit of $f(x)$ as x approaches a from the left (or from below). For example, the function $\mathrm{sgn} : \mathbf{R} \to \mathbf{R}$ defined by

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$

satisfies $\mathrm{sgn}(0) = 0$, $\lim_{x \to 0^+} \mathrm{sgn}(x) = 1$ and $\lim_{x \to 0^-} \mathrm{sgn}(x) = -1$. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

   (a) What are $\lim_{x \to 0^+} \frac{1}{x}$ and $\lim_{x \to 0^-} \mathrm{sgn}(x^2)$? Is $f(x) = \mathrm{sgn}(x^2)$ continuous at $x = 0$?

   (b) Suppose $g(x) = \begin{cases} x^2 + ax + 1, & x < 3, \\ ax - x^2, & x \geq 3. \end{cases}$ For what values of a is $g(x)$ a continuous function? In this case, is g differentiable at $x = 3$?

2. Compute $\lim_{h \to 0} \dfrac{\sin\left(\frac{\pi}{6} + h\right) - \sin \frac{\pi}{6}}{h}$.

3. Compute $\frac{d}{dx}(x-2)^{-2}$.

4. If u is a differentiable function of x, $u'(2) = 5$ and $u(x) \cdot v(x)^2 = x$ for all x, compute $v'(2)$ in terms of $v(2)$.

5. Compute $\frac{d}{dx}\left(\sin^2 x + \cos^2 x\right)$.

6. What is the equation of the tangent line to $y = x^2 + 1$ at the point $(t, t^2 + 1)$?

7. Suppose that $\frac{x+y}{x-y} = x^2$. Find $\frac{dy}{dx}$ by implicit differentiation and also by solving for y and then differentiating. Verify that your two answers are the same.

8. Find the maximal possible volume of a right cone inscribed in a sphere of radius R.

9. Draw the graph of $y = x^3 - 6x^2$. Mark all local extrema and inflection points.

10. Compute $\frac{d}{dx} \ln(x^2)$ and $\frac{d}{dx} \ln(\ln x)$.

11. Define the functions $\cosh x = \frac{e^x+e^{-x}}{2}$ and $\sinh x = \frac{e^x-e^{-x}}{2}$. Show that $\cosh^2 x - \sinh^2 x = 1$ and compute $\frac{d}{dx}\cosh x$ and $\frac{d}{dx}\sinh x$. What is $\frac{d^{13}}{dx^{13}}\cosh x|_{x=0}$?

12. A line L is tangent to the curve $y = x^4+bx^3+cx^2-4x+3$ at $x=1$ and at $x=-1$. Compute b and c.

13. What is the tangent line to $y=\sqrt{1+x}$ at $x=0$? (This is a common approximation used when x is near 0.)

14. For each of the following antiderivatives, recognize the appropriate substitution. Choose three and compute the antiderivative (using the selected substitution or any other method).

    (a) $\int \sqrt{3+4x}\, dx$
    (b) $\int 3x^2(x^3)^4\, dx$
    (c) $\int xe^{-x^2}\, dx$
    (d) $\int \frac{dx}{x\ln x}$
    (e) $\int \frac{x\, dx}{\sqrt{9-4x^2}}$
    (f) $\int \frac{dx}{\sqrt{9-4x^2}}$
    (g) $\int \cos(x)(1+\sin x)^6\, dx$
    (h) $\int \frac{dx}{\sqrt{x}\cdot\sqrt{\sqrt{x}+1}}$

15. Use the substitution $x=\cosh u$ to integrate $\int \frac{dx}{\sqrt{x^2-1}}$ (see problem 11). You may leave your answer in terms of the functional inverse of the cosh function.

16. Compute the area between two consecutive intersections of $y=\sin x$ and $y=\cos x$.

17. The vertical line $x=a$ divides the region in the first quadrant bounded by the x-axis and $y=x-x^3$ into two pieces of equal area. Compute a.

18. Assume $u>0$. Compute the area in the first quadrant bounded by the x-axis, the hyperbola $x^2-y^2=1$ and the line $y=\frac{\sinh u}{\cosh u}x$. (See problem 11.)

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19. Place the following functions in "asymptotic size order," using the following operations: write $f(x) \gg g(x)$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$. Write $f(x)>g(x)$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=L>1$. Write $f(x)\sim g(x)$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$. Write $f(x)=g(x)$ if actually f and g are equal for all values of x. (Note that other possibilities exist, but do not occur among the functions listed here.)

    (a) e^{x^2}
    (b) $e^{x\ln x}$
    (c) $e^{(\ln x)^2}$
    (d) $e^{\ln x}$
    (e) x
    (f) x^2

(g) x^3

(h) x^x

(i) $x \ln x$

(j) $\ln(e^x + x)$

(k) $\ln(x)^3$

(l) $\ln(x^3)$

(m) $x^3 + e^x$

(n) $\ln x \cdot e^x$

(o) $\ln(x + \sin x)$

(p) 1

(q) $\frac{1}{x}$

(r) e^{-x}

20. Use polynomial long division, completing the square and a trig substitution to compute $\int \frac{2x^3+6x^2+6x+3}{x^2+x+\frac{1}{2}}\,dx$.

21. In class, we used a clever trick to compute $\int \frac{2}{x^2-1}\,dx$ by writing $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$. Use a similar trick (not a trig substitution) to compute $\int \frac{1}{x^2+6x+8}\,dx$: first factor the denominator as $(x+a)(x+b)$, then find constants A and B so that $\frac{1}{x^2+6x+8} = \frac{A}{x+a} + \frac{B}{x+b}$. Then integrate.

22. Compute $\int_e^{e^2} \frac{1}{x \ln x}\,dx$.

23. Compute $\int_{-1}^{1} xe^{-x^2}\,dx$.

24. Sketch the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ and compute its total arclength. (Suggestion: if (a, b) is on the curve then so are $(a, -b)$, $(-a, b)$ and $(-a, -b)$. This means you can graph it just in the first quadrant and then use symmetry.)

25. Compute the length of the curve $y = \ln \sin x$ for $\frac{\pi}{6} \leq x \leq \frac{\pi}{4}$.

26. Use partial fractions to compute the antiderivative $\int \frac{1}{x^2(x-1)^2(x^2+1)}\,dx$. Use the antiderivative to compute the definite integral $\int_2^3 \frac{1}{x^2(x-1)^2(x^2+1)}\,dx$.

27. Compute the volume of the solid of revolution formed by rotating the first-quadrant region between $y = \sqrt{x}$ and $y = x^2$ around the y-axis.

28. (a) Let S be the square region bounded between the x-axis and the line $y = 1$ and between the vertical lines $x = 2$ and $x = 3$. Use three methods (Pappus' Theorem, the washer method and the shell method) to compute the volume of the solid that results when you rotate region S around the y-axis.

    (b) Let R be the region under the curve $y = \sqrt{4 - x^2}$ for $1 \leq x \leq 2$. This region is rotated around the x-axis to form a solid (a "cap" of a sphere). Write down two integrals (one from the disk method and one from the shell method) for the volume of this solid. Choose one of them to evaluate.

29. Recall the graph of the function $y = \arccos x$ from lecture 2. Write down two integrals that give the area between this curve and the x-axis, one with respect to x and the other with respect to y. Choose one to evaluate.

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30. Use integration by parts to compute the following integrals.

(a) $\int x^2 \sin x \, dx$

(b) $\int x \ln(x) \, dx$

(c) $\int \arcsin x \, dx$

(d) $\int_0^\pi x \sin x \, dx$

31. Compute $\frac{d}{dx}\pi^x$ and $\int e^{2x} \, dx$.

32. Recall the Fundamental Theorems of Calculus, and use it to compute the following integrals. (Be careful! Only the first is straight-forward.)

(a) $\frac{d}{dx}\int_0^x te^t \, dt$

(b) $\frac{d}{dx}\int_0^{x^2} \sin(t^3) \, dt$

(c) $\frac{d}{dx}\int_{-x}^{x} g(t) \, dt$

33. Compute the following integrals. Some tools that may be helpful: a well-chosen substitution; the Pythagoran identity; the double angle formulas.

(a) $\int \cos^2 x \, dx$

(b) $\int \cos^2 x \sin x \, dx$

(c) $\int \cos^3 x \sin^3 x \, dx$

(d) $\int \cos^2 x \sin^2 x \, dx$

34. Which of the following improper integrals converge, and why?

(a) $\int_0^\infty \frac{x^2}{x^4+1} \, dx$

(b) $\int_0^\infty \frac{x^3}{x^4+1} \, dx$

(c) $\int_1^\infty \frac{1}{te^t} \, dt$

(d) $\int_0^1 \frac{1}{te^t}\,dt$

35. Use the comparison or limit comparison test to say whether the following sums converge.

(a) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

(c) $\sum_{n=1}^{\infty} \frac{3n+2}{n+1} \cdot \frac{4^n}{5^n - 1}$

36. Let $A = \langle 3, 1, 2 \rangle$, $B = \langle -1, 0, 1 \rangle$ and $C = \langle 1, 2, 1 \rangle$ be three-dimensional vectors. Compute all of the following: $A+B$, $C-2A$, $B \cdot A$, $A \cdot C$, $A \times B$, $A \cdot (B \times C)$, $(A \times B) \times C$, $A \times (B \times C)$.

37. Let $\mathbf{A}$ and $\mathbf{B}$ be two-dimensional vectors. If the 2×2 determinant $\det \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = 0$, what does this say (geometrically) about $\mathbf{A}$ and $\mathbf{B}$?

38. Let $M_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ and $M_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -2 \end{bmatrix}$. There are nine possible products $M_i \cdot M_j$ for $1 \le i \le 3$, $1 \le j \le 3$, only five of which are legal matrix multiplications. Identify and compute these five products.

39. Of the five matrices that resulted from the previous problem, three are square. Compute the determinants of these matrices. For those that are invertible, compute their inverses.

40. Let $\mathbf{V} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{W} = c(\mathbf{j} + \mathbf{k})$. For which value of c are $\mathbf{V}$ and $\mathbf{W}$ perpendicular? Is there any value of c for which $\mathbf{V}$ and $\mathbf{W}$ are parallel? (Why or why not?)

41. What is the angle between two different diagonals of a cube? (Hint: it may help to choose some nice coordinates for the vertices. And, of course, the answer doesn't depend on the side length.)

42. What is the inverse of the matrix $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$?

43. Write the system of equations

$$\begin{cases} 2x + y = 3 \\ x - 2y = 1 \end{cases}$$

as a matrix equation (i.e., some matrix times some column vector equals some other column vector). Solve it by finding the inverse of the appropriate matrix (not!! by other methods).

44. For what values of c are the vectors $\mathbf{V} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{W} = c\mathbf{i} + (3 - 2c)\mathbf{j} + \mathbf{k}$ parallel? Perpendicular?

45. Use the integral test to say whether the following series converge. (Note that the unusual initial values in the second and third examples are just there so that the series really are decreasing and positive.)

   (a) $\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2}$
   (b) $\sum_{n=3}^{\infty} \frac{1}{n \cdot (\ln n) \cdot \ln(\ln n)}$
   (c) $\sum_{n=3}^{\infty} \frac{1}{n \cdot (\ln n) \cdot (\ln(\ln n))^2}$

46. Use the root or ratio test to say whether the following series converge.

   (a) $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$
   (b) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$
   (c) $\sum_{n=0}^{\infty} \frac{2^{3n}}{3^{2n}}$

47. Write down one example each of a convergent and divergent alternating series. Think of something we haven't done in class.

48. Compute the first four nonzero terms of the power series for the following functions, using a method of your choice:

   (a) $\frac{1}{1+x}$
   (b) $\ln(1+x)$
   (c) e^{x^2}
   (d) $\sin^2 x$
   (e) $\frac{1}{1+x^2}$
   (f) $\arctan x$

49. Using your knowledge of power series from class, compute the values of the following numerical series.

   (a) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$
   (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$ (Hint: use part (f) of the previous problem.)
   (c) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$ (Hint: use one of the power series I marked as "important" in class, or part (b) of the previous problem.)