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Exercise Sheet 3

Out Fri 03/09/07
Due Fri 03/23/07

Exercise 5 Consider the Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

1. Using your knowledge about Airy's equation $u_t + u_{xxx} = 0$, explain how solutions to the KdV equation are expected to differ from solution to Burgers' equation $u_t + 6uu_x = 0$.
2. Write a Matlab program which approximates the KdV equation by finite differences. Consider the interval $[-1, 1]$ with periodic boundary conditions. I suggest an explicit step for the nonlinear advection and an implicit step for the dispersion term, but you are free to use other methods. In order to get a reasonable resolution you should use at least 300 grid points.
3. Define the function
$$f_c(x) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}x\right)$$
where $\operatorname{sech}(x) = \frac{2}{e^x+e^{-x}}$. If you like differentiating, you can verify (as a private exercise) that for every $c > 0$ indeed $u(x,t) = f_c(x-ct)$ is a solution to the KdV equation (such travelling waves are called *solitons*). Now run your program with the following initial data
   (a) $u_0(x) = f_{400}(x)$
   (b) $u_0(x) = f_{400}(x+0.7) + f_{200}(x)$
   (c) $u_0(x) = \frac{1}{2}\left(f_{400}(x+0.7) + f_{200}(x)\right)$

   Plot the results at time $t = 0.015$. Explain briefly how the three cases behave.

Exercise 6 Download the matlab file `mit18086_levelsetmethod.m` on the course web page, and run it with `mit18086_levelsetmethod(1)` and `mit18086_levelsetmethod(2)`.

1. Explain what the two given velocity fields do to the initial geometry.
2. Change the code to model a bush fire (with the same initial geometry), spreading with velocity 1. Is the domain of fire connected at time $t = 2$? Provide a numerical result which shows the correct topology.

Exercise 7 Download and run the matlab file `mit18086_navierstokes.m` on the course web page.

1. For Reynolds numbers of 0.1, 100, and 5000, run the simulation long enough, until a steady state is achieved. Describe the differences in the three solutions. Look in particular at symmetries and the flow near the two lower corners. You might want to increase the resolution and visualize the velocity field in a better way to observe the behavior.
2. Which resolution would you need to resolve the smallest scales in the given problem for the above Reynolds numbers?
3. Change the boundary conditions to a clockwise flow parallel to the walls which is 1 at the top, 2 at the right, 3 at the bottom, and 4 at the left. Compare the stationary solutions for low and large Reynolds numbers.