Notation and Nomenclature

$\mathbf{A}$	Matrix
$\mathbf{A}_{ij}$	Matrix indexed for some purpose
$\mathbf{A}_i$	Matrix indexed for some purpose
$\mathbf{A}^{ij}$	Matrix indexed for some purpose
$\mathbf{A}^n$	Matrix indexed for some purpose **or**
	The n.th power of a square matrix
$\mathbf{A}^{-1}$	The inverse matrix of the matrix $\mathbf{A}$
$\mathbf{A}^+$	The pseudo inverse matrix of the matrix $\mathbf{A}$ (see Sec. 3.6)
$\mathbf{A}^{1/2}$	The square root of a matrix (if unique), not elementwise
$(\mathbf{A})_{ij}$	The (i,j).th entry of the matrix $\mathbf{A}$
A_{ij}	The (i,j).th entry of the matrix $\mathbf{A}$
$[\mathbf{A}]_{ij}$	The ij-submatrix, i.e. $\mathbf{A}$ with i.th row and j.th column deleted
$\mathbf{a}$	Vector (column-vector)
$\mathbf{a}_i$	Vector indexed for some purpose
a_i	The i.th element of the vector $\mathbf{a}$
a	Scalar
$\Re z$	Real part of a scalar
$\Re \mathbf{z}$	Real part of a vector
$\Re \mathbf{Z}$	Real part of a matrix
$\Im z$	Imaginary part of a scalar
$\Im \mathbf{z}$	Imaginary part of a vector
$\Im \mathbf{Z}$	Imaginary part of a matrix
$\det(\mathbf{A})$	Determinant of $\mathbf{A}$
$\mathrm{Tr}(\mathbf{A})$	Trace of the matrix $\mathbf{A}$
$\mathrm{diag}(\mathbf{A})$	Diagonal matrix of the matrix $\mathbf{A}$, i.e. $(\mathrm{diag}(\mathbf{A}))_{ij} = \delta_{ij} A_{ij}$
$\mathrm{eig}(\mathbf{A})$	Eigenvalues of the matrix $\mathbf{A}$
$\mathrm{vec}(\mathbf{A})$	The vector-version of the matrix $\mathbf{A}$ (see Sec. 10.2.2)
$\sup$	Supremum of a set
$\|\|\mathbf{A}\|\|$	Matrix norm (subscript if any denotes what norm)
$\mathbf{A}^T$	Transposed matrix
$\mathbf{A}^{-T}$	The inverse of the transposed and vice versa, $\mathbf{A}^{-T} = (\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$.
$\mathbf{A}^*$	Complex conjugated matrix
$\mathbf{A}^H$	Transposed and complex conjugated matrix (Hermitian)
$\mathbf{A} \circ \mathbf{B}$	Hadamard (elementwise) product
$\mathbf{A} \otimes \mathbf{B}$	Kronecker product
$\mathbf{0}$	The null matrix. Zero in all entries.
$\mathbf{I}$	The identity matrix
$\mathbf{J}^{ij}$	The single-entry matrix, 1 at (i,j) and zero elsewhere
$\mathbf{\Sigma}$	A positive definite matrix
$\mathbf{\Lambda}$	A diagonal matrix

1 Basics

$$
\begin{aligned}
(\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} && (1)\\
(\mathbf{ABC}...)^{-1} &= ...\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} && (2)\\
(\mathbf{A}^T)^{-1} &= (\mathbf{A}^{-1})^T && (3)\\
(\mathbf{A}+\mathbf{B})^T &= \mathbf{A}^T+\mathbf{B}^T && (4)\\
(\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T && (5)\\
(\mathbf{ABC}...)^T &= ...\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T && (6)\\
(\mathbf{A}^H)^{-1} &= (\mathbf{A}^{-1})^H && (7)\\
(\mathbf{A}+\mathbf{B})^H &= \mathbf{A}^H+\mathbf{B}^H && (8)\\
(\mathbf{AB})^H &= \mathbf{B}^H\mathbf{A}^H && (9)\\
(\mathbf{ABC}...)^H &= ...\mathbf{C}^H\mathbf{B}^H\mathbf{A}^H && (10)
\end{aligned}
$$

1.1 Trace

$$
\begin{aligned}
\mathrm{Tr}(\mathbf{A}) &= \textstyle\sum_i A_{ii} && (11)\\
\mathrm{Tr}(\mathbf{A}) &= \textstyle\sum_i \lambda_i, \qquad \lambda_i = \mathrm{eig}(\mathbf{A}) && (12)\\
\mathrm{Tr}(\mathbf{A}) &= \mathrm{Tr}(\mathbf{A}^T) && (13)\\
\mathrm{Tr}(\mathbf{AB}) &= \mathrm{Tr}(\mathbf{BA}) && (14)\\
\mathrm{Tr}(\mathbf{A}+\mathbf{B}) &= \mathrm{Tr}(\mathbf{A})+\mathrm{Tr}(\mathbf{B}) && (15)\\
\mathrm{Tr}(\mathbf{ABC}) &= \mathrm{Tr}(\mathbf{BCA}) = \mathrm{Tr}(\mathbf{CAB}) && (16)\\
\mathbf{a}^T\mathbf{a} &= \mathrm{Tr}(\mathbf{a}\mathbf{a}^T) && (17)
\end{aligned}
$$

1.2 Determinant

Let $\mathbf{A}$ be an $n \times n$ matrix.

$$
\begin{aligned}
\det(\mathbf{A}) &= \textstyle\prod_i \lambda_i \qquad \lambda_i = \mathrm{eig}(\mathbf{A}) && (18)\\
\det(c\mathbf{A}) &= c^n \det(\mathbf{A}), \qquad \text{if } \mathbf{A} \in \mathbb{R}^{n\times n} && (19)\\
\det(\mathbf{A}^T) &= \det(\mathbf{A}) && (20)\\
\det(\mathbf{AB}) &= \det(\mathbf{A})\det(\mathbf{B}) && (21)\\
\det(\mathbf{A}^{-1}) &= 1/\det(\mathbf{A}) && (22)\\
\det(\mathbf{A}^n) &= \det(\mathbf{A})^n && (23)\\
\det(\mathbf{I}+\mathbf{u}\mathbf{v}^T) &= 1+\mathbf{u}^T\mathbf{v} && (24)
\end{aligned}
$$

For $n = 2$:

$$\det(\mathbf{I}+\mathbf{A}) = 1+\det(\mathbf{A})+\mathrm{Tr}(\mathbf{A}) \qquad (25)$$

For $n = 3$:

$$\det(\mathbf{I}+\mathbf{A}) = 1+\det(\mathbf{A})+\mathrm{Tr}(\mathbf{A})+\frac{1}{2}\mathrm{Tr}(\mathbf{A})^2-\frac{1}{2}\mathrm{Tr}(\mathbf{A}^2) \qquad (26)$$

For $n = 4$:

$$\begin{aligned}\det(\mathbf{I}+\mathbf{A}) &= 1+\det(\mathbf{A})+\mathrm{Tr}(\mathbf{A})+\frac{1}{2} \\ &\quad +\mathrm{Tr}(\mathbf{A})^2-\frac{1}{2}\mathrm{Tr}(\mathbf{A}^2) \\ &\quad +\frac{1}{6}\mathrm{Tr}(\mathbf{A})^3-\frac{1}{2}\mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\mathbf{A}^2)+\frac{1}{3}\mathrm{Tr}(\mathbf{A}^3)\end{aligned} \tag{27}$$

For small ε, the following approximation holds

$$\det(\mathbf{I}+\varepsilon\mathbf{A}) \cong 1+\det(\mathbf{A})+\varepsilon\mathrm{Tr}(\mathbf{A})+\frac{1}{2}\varepsilon^2\mathrm{Tr}(\mathbf{A})^2-\frac{1}{2}\varepsilon^2\mathrm{Tr}(\mathbf{A}^2) \tag{28}$$

1.3 The Special Case 2x2

Consider the matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Determinant and trace

$$\det(\mathbf{A}) = A_{11}A_{22} - A_{12}A_{21} \tag{29}$$

$$\mathrm{Tr}(\mathbf{A}) = A_{11} + A_{22} \tag{30}$$

Eigenvalues

$$\lambda^2 - \lambda \cdot \mathrm{Tr}(\mathbf{A}) + \det(\mathbf{A}) = 0$$

$$\lambda_1 = \frac{\mathrm{Tr}(\mathbf{A}) + \sqrt{\mathrm{Tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}}{2} \qquad \lambda_2 = \frac{\mathrm{Tr}(\mathbf{A}) - \sqrt{\mathrm{Tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}}{2}$$

$$\lambda_1 + \lambda_2 = \mathrm{Tr}(\mathbf{A}) \qquad \lambda_1\lambda_2 = \det(\mathbf{A})$$

Eigenvectors

$$\mathbf{v}_1 \propto \begin{bmatrix} A_{12} \\ \lambda_1 - A_{11} \end{bmatrix} \qquad \mathbf{v}_2 \propto \begin{bmatrix} A_{12} \\ \lambda_2 - A_{11} \end{bmatrix}$$

Inverse

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \tag{31}$$