18.085: More on Fourier Series

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0.1 Complex Fourier Series

We defined the Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx). \tag{1}$$

We can rewrite it in a more compact form if we use the identities

$$\cos(nx) = \frac{1}{2}(e^{inx} + e^{-inx}), \tag{2}$$

$$\sin(nx) = \frac{1}{2i}(e^{inx} - e^{-inx}). \tag{3}$$

We get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - ib_n)e^{inx} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n + ib_n)e^{-inx}. \tag{4}$$

Now define

$$c_0 = \frac{a_0}{2}, \tag{5}$$

$$c_n = \frac{a_n - ib_n}{2} \quad \text{if } n > 0, \tag{6}$$

$$c_n = \frac{a_n + ib_n}{2} \quad \text{if } n < 0. \tag{7}$$

With this new notation we arrive at the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{8}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx. \tag{9}$$

0.2 Four properties of Fourier series

From now on, we will stick to the complex version of Fourier series.

Property 1. Energy (or norm) of a (2π periodic) function is defined as

$$\text{energy in } f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx. \tag{10}$$

Suppose we know the Fourier coefficients of $f(x)$. As you will show in homework, the energy can be equivalently written as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{11}$$

The above identity is called the *Plancherel's identity*. It says that the energy in x-space equals the energy in Fourier space. It is essentially the Pythagorean theorem for functions.

Property 2. If we know the Fourier coefficients of $f(x)$ what are the Fourier coefficients of its derivative? If we differentiate the Fourier series of $f(x)$ term by term we get

$$\frac{df}{dx} = \sum_{n=-\infty}^{\infty} c_n in e^{inx}. \tag{12}$$

We conclude that $\frac{df}{dx}$ has Fourier coefficients inc_n. Differentiation of functions becomes multiplication in Fourier space. Note that differentiation puts more weight on higher frequencies.

Property 3. What if we integrate $f(x)$? If $c_0 = 0$, term by term integration gives

$$\int f(x)dx = \sum_{n=-\infty}^{\infty} \frac{c_n}{in} e^{inx}. \tag{13}$$

We conclude that if $c_0 = 0$, $\int f$ has Fourier coefficients $\frac{c_n}{in}$. Note that integration puts more weight on low frequencies.

Property 4. What if we simply perform the shift $f(x-s)$? We get

$$f(x-s) = \sum_{n=-\infty}^{\infty} c_n e^{in(x-s)} = \sum_{n=-\infty}^{\infty} (c_n e^{-ins}) e^{inx}. \tag{14}$$

The shift has Fourier coefficients $e^{-ins}c_n$. Note that this transformation doesn't affect the energy in $f(x)$.

0.3 Fifth property of Fourier series - multiplication and convolution

Suppose we know the Fourier coefficients, c_n and d_n, of two functions, $f(x)$ and $g(x)$. What are the Fourier coefficients of the product $f(x)g(x)$? The answer is

called the *discrete convolution* and is denoted by $(c \star d)_n$. Let's work out the expression for $(c * d)_n$. By definition

$$f(x)g(x) = \sum_k \sum_l c_k d_l e^{i(k+l)x} = \sum_k (c * d)_n e^{inx}. \tag{15}$$

If we compare the two sides we get

$$(c * d)_n = \sum_k c_k d_{n-k}. \tag{16}$$

The results might slightly disappoint you if you were hoping for a simple result such as $c_k d_k$. The above shows that the Fourier coefficients of the product of functions is *not* a product of their coefficients. Instead, it is the discrete convolution.

The related question is: what operation on $f(x)$ and $g(x)$ has Fourier coefficients that is just the product of the coefficients of f and g? The answer is the *periodic convolution*. For periodic functions f and g, it is defined as

$$(f * g)(x) = \int_0^{2\pi} f(t)g(x-t)dt. \tag{17}$$

In the homework you will show that $(f * g)(x)$ has Fourier coefficients $2\pi c_k d_k$.

0.4 2D Fourier Transform

We can generalize the Fourier transform to function of two (and more) variables f(x,y). The transform is

$$f(x,y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm} e^{inx} e^{imy}. \tag{18}$$

The Fourier coefficients are

$$c_{mn} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) e^{-imx} e^{-iny} dxdy. \tag{19}$$