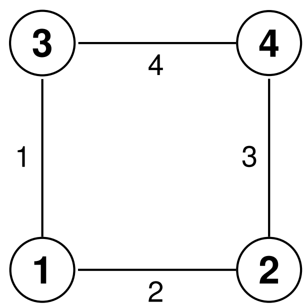
18.085: EXAM 2 SOLUTIONS

July 28, 2014

Question 1. (25 pts.)

Consider the following plane square truss:



(a) (5 pts.) Write down the matrix A for this truss.

Solution: There are 4 nodes and 4 edges, so the matrix A should be 4×8. Each row corresponds to an edge, and each node has two displacements u_i^H (horizontal) and u_i^V (vertical). The matrix A thus looks like

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

(b) (3 pts.) What is the rank of A?

Hint: It is probably easiest to determine the number of independent rows.

Solution: It is clear by inspection that all four rows of A are independent, so $\text{rank}(A) = 4$.

(c) (5 pts.) How many instabilities (rigid motions and mechanisms) does this truss have? Write down the corresponding displacement vector(s) $\vec{u}$.

Hint: Recall from earlier in the course that

$$\text{rank}(A) + (\text{number of independent solutions to } A\vec{u} = \vec{0}) = \text{number of columns in } A$$

Solution: We know from part (b) that rank$(A) = 4$. Since A has 8 columns, we can use the formula in the hint to deduce that $A\vec{u} = \vec{0}$ must have 4 independent solutions. Thus, the truss has four different instabilities. They correspond to: (i.) rigid horizontal motion, (ii.) rigid vertical motion, (iii.) moving nodes 3 and 4 horizontally, and (iv.) moving nodes 2 and 4 vertically. The first two are rigid motions, and the last two are mechanisms. The corresponding displacement vectors $\vec{u}$ are

(i.) $\vec{u} = \begin{pmatrix} d & 0 & d & 0 & d & 0 & d & 0 \end{pmatrix}^T$

(ii.) $\vec{u} = \begin{pmatrix} 0 & d & 0 & d & 0 & d & 0 & d \end{pmatrix}^T$

(iii.) $\vec{u} = \begin{pmatrix} 0 & 0 & 0 & 0 & d & 0 & d & 0 \end{pmatrix}^T$

(iv.) $\vec{u} = \begin{pmatrix} 0 & 0 & 0 & d & 0 & 0 & 0 & d \end{pmatrix}^T$

It is easy to verify that $A\vec{u} = \vec{0}$ for each of these vectors.

(d) (3 pts.) Now fix nodes 1 and 2. Write down the matrix A_1 corresponding to this new truss.

Solution: We remove the first four columns of the matrix A derived in part (a), since nodes 1 and 2 are fixed. We thus obtain

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

(e) (4 pts.) What is the rank of A_1, and how many instabilities does the truss have? Write down the corresponding displacement vector(s) $\vec{u}$.

Solution: One of the rows of A_1 is all zeros, and the remaining rows are clearly independent, so rank$(A_1) = 3$. Using the formula from the hint in part (c). we conclude that there is one solution to $A_1\vec{u} = \vec{0}$. The truss thus has one instability, which corresponds to a horizontal translation of nodes 3 and 4. The corresponding vector $\vec{u}$ is

$$\vec{u} = \begin{pmatrix} d & 0 & d & 0 \end{pmatrix}^T$$

(f) (5 pts.) For the truss in part (d) (with nodes 1 and 2 fixed), add a bar between nodes 1 and 4. Write down the new matrix A_2. How many mechanisms does this truss have? Explain your answer mathematically; do not give purely physical arguments!

Solution: The bar from nodes 1 and 4 will make an angle of 45° with the horizontal. Since $\cos(45°) = \sin(45°) = 1/\sqrt{2}$, we find that

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Clearly the rank of A_2 is 4, since there are four independent rows. Since A_2 also has four columns, there must be no solutions to $A\vec{u} = \vec{0}$, and thus the truss has no mechanisms.

Question 2. (10 pts.)

Consider the equation

$$x^3 + 4 = 2x$$

Use Newton's method to find an approximate solution to this equation. Specifically, carry out two iterations of Newton's method starting with the initial guess $x_0 = 0$.

Solution: We want to find a root of the polynomial $f(x) = x^3 - 2x + 4$, for which $f'(x) = 3x^2 - 2$. Newton's method says that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Starting at $x_0 = 0$, we have

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{4}{-2} = 2 \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{8}{10} = 1.2 \end{aligned}$$

Question 3. (25 pts.)

We consider a non-uniform beam of length 2 hanging under the force of gravity. Specifically, we will solve the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(c(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right) = f(x), \quad 0 \leq x \leq 2$$

The force is $f(x) = -1$, and the stiffness function $c(x)$ is

$$c(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

We assume the beam to be simply supported, so we take the boundary conditions

$$u = 0, \quad M = 0$$

at both ends, where $M(x) = c(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$ is the bending moment.

(a) (10 pts.) Find the bending moment $M(x) = c(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}$. (Don't forget to impose the boundary conditions on M.)

Solution: Since $f(x) = -1$, the bending moment $M(x)$ solves the equation

$$M'' = -1, \quad M(0) = M(2) = 0$$

Integrating the equation twice, we have

$$M' = -x + A, \quad M = -\frac{x^2}{2} + Ax + B$$

The condition $M(0) = 0$ implies that $B = 0$. The condition $M(2) = 0$ implies that $-2 + 2A = 0$, or that $A = 1$. The bending moment is thus

$$M = -\frac{x^2}{2} + x$$

(b) (5 pts.) Find u''.

Solution: Since $u'' = M(x)/c(x)$, we find that

$$u''(x) = \begin{cases} -\frac{x^2}{2} + x & \text{if } x \leq 1 \\ -\frac{x}{2} + 1 & \text{if } x > 1 \end{cases}$$

(c) (10 pts.) Find the displacement $u(x)$ by imposing the boundary conditions on $u(x)$. Also impose the requirement that u and u' must be continuous along the beam. Once you impose these conditions, you should obtain equations that determine the constants in $u(x)$. **You do not have to solve the equations.**

Solution: Integrating the expression for u'' twice, we obtain

$$\begin{aligned} u'(x) &= \begin{cases} -\frac{x^3}{6} + \frac{x^2}{2} + C & \text{if } x \leq 1 \\ -\frac{x^2}{4} + x + D & \text{if } x > 1 \end{cases} \\ u(x) &= \begin{cases} -\frac{x^4}{24} + \frac{x^3}{6} + Cx + E & \text{if } x \leq 1 \\ -\frac{x^3}{12} + \frac{x^2}{2} + Dx + F & \text{if } x > 1 \end{cases} \end{aligned}$$

Since $u(0) = 0$, we find that $E = 0$. Since $u(2) = 0$, we find that

$$-\frac{2}{3} + 2 + 2D + F = 0 \Rightarrow \boxed{2D + F = -\frac{4}{3}}$$

We now impose the conditions that u and u' are continuous along the beam. That is, we need to ensure that the piecewise functions defining $u'(x)$ and $u(x)$ agree at $x = 1$. The condition on u' implies that

$$-\frac{1}{6} + \frac{1}{2} + C = -\frac{1}{4} + 1 + D \Rightarrow \boxed{C - D = \frac{5}{12}}$$

The condition on u implies that

$$-\frac{1}{24} + \frac{1}{6} + C \quad = \quad -\frac{1}{12} + \frac{1}{2} + D + F \Rightarrow \boxed{C - D - F = \frac{7}{24}}$$

The three boxed equations determine the constants C, D and F.

You did not have to find the constants. If we solve the equations, we find

$$C = -\frac{5}{16}, \quad D = -\frac{35}{48}, \quad F = \frac{1}{8}$$

Question 4. (15 pts.)

Solve Laplace's equation $\Delta u = 0$ on the unit disc with the boundary condition

$$\frac{\partial u}{\partial r} = \cos 2\theta - \sin 4\theta$$

on the boundary of the disc.

Solution: We showed in class that $r^n \cos(n\theta)$ and $r^n \sin(n\theta)$ are solutions to Laplace's equation. In order to satisfy the boundary condition, we need to take functions of the form $r^2 \cos(2\theta)$ and $r^4 \sin(4\theta)$. Specifically, the solution is

$$u(r, \theta) = \frac{r^2}{2} \cos 2\theta - \frac{r^4}{4} \sin 4\theta$$

You can verify that $u(r, \theta)$ satisfies the boundary condition at $r = 1$.

Question 5. (25 pts.) In this question, we will set up the finite element method for the equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(c(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = \delta(x-2/3), \quad 0 \le x \le 1$$

where

$$c(x) = \begin{cases} 2 & \text{if } x < 1/3 \\ 4 & \text{if } x > 1/3 \end{cases}$$

We take the boundary conditions

$$w(0) = u(1) = 0$$

where $w(x) = c(x)\frac{\mathrm{d}u}{\mathrm{d}x}$.

(a) (5 pts.) Write down the weak form of the differential equation. What condition must the test functions $v(x)$ satisfy?

Solution: We multiply both sides of the equation by a test function $v(x)$ and integrate from 0 to 1.

$$\int_0^1 -\frac{\mathrm{d}}{\mathrm{d}x}\left(c(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) v(x)\,\mathrm{d}x = \int_0^1 \delta(x-2/3)v(x)\,\mathrm{d}x$$

We integrate the left-hand side by parts:

$$\begin{aligned}\int_0^1 -\frac{\mathrm{d}}{\mathrm{d}x}\left(c(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) v(x)\,\mathrm{d}x &= c(x)\frac{\mathrm{d}u}{\mathrm{d}x}v(x)\Big|_{x=0}^{x=1} + \int_0^1 c(x)\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x \\ &= \int_0^1 c(x)\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x\end{aligned}$$

The boundary conditions imply that $w = cu' = 0$ at $x = 0$ and the test function satisfies $v(1) = 0$. We thus lose the boundary term in the equation above, and obtain the weak form

$$\int_0^1 c(x)\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int_0^1 \delta(x-2/3)v(x)\,\mathrm{d}x$$

(b) (5 pts.) Take $h = 1/3$. Draw the hat functions you will use to solve this problem.

Solution: We use the hat functions ϕ_0, ϕ_1 and ϕ_2; we omit ϕ_3 because it does not satisfy the boundary condition $u(1) = 0$. The hat functions are shown in Figure 1.

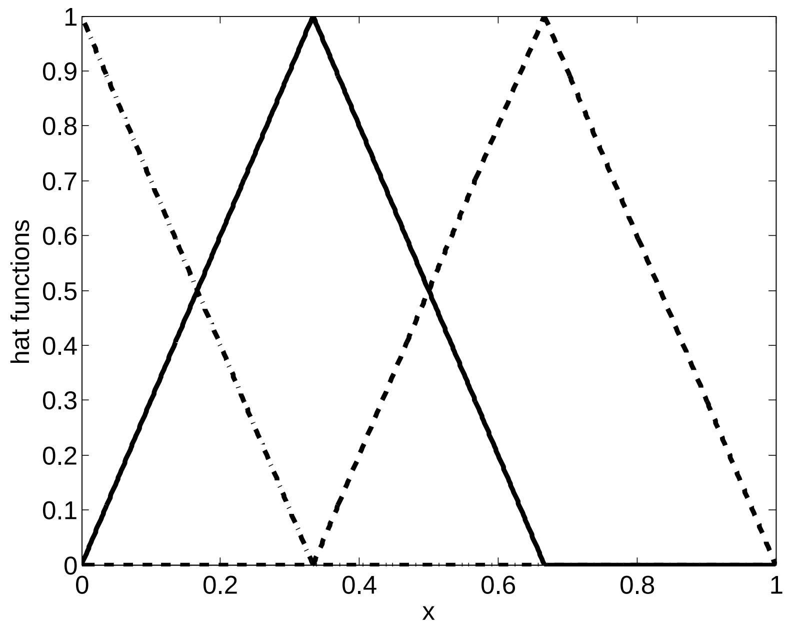


FIGURE 1. Solution to Question 5(b). The hat functions are ϕ_0 (dot-dash line), ϕ_1 (solid line) and ϕ_2 (dashed line).

(c) (10 pts.) Construct the matrix K for this problem. Show your work.

Solution: The matrix K has matrix elements

$$\int_0^1 c(x)\phi_i'(x)\phi_j'(x)\,\mathrm{d}x$$

The derivatives of ϕ' are ± 3 on the appropriate intervals, so these integrals are relatively straightforward. We only need to remember to account for the fact that $c(x)$ changes at $x = 1/3$.

$$\begin{aligned}
K_{11} &= \int_0^1 c(x)(\phi_0')^2\,\mathrm{d}x = \int_0^{1/3} 2\cdot(-3)^2\,\mathrm{d}x = 6 \\
K_{22} &= \int_0^1 c(x)(\phi_1')^2\,\mathrm{d}x = \int_0^{1/3} 2\cdot 3^2\,\mathrm{d}x + \int_{1/3}^{2/3} 4\cdot(-3)^2\,\mathrm{d}x = 18 \\
K_{33} &= \int_0^1 c(x)(\phi_2')^2\,\mathrm{d}x = \int_{1/3}^{2/3} 4\cdot 3^2\,\mathrm{d}x + \int_{2/3}^{1} 4\cdot(-3)^2\,\mathrm{d}x = 24 \\
K_{12} &= \int_0^1 c(x)\phi_0'(x)\phi_1'(x)\,\mathrm{d}x = \int_0^{1/3} 2\cdot 3\cdot(-3)\,\mathrm{d}x = -6 \\
K_{23} &= \int_0^1 c(x)\phi_1'(x)\phi_2'(x)\,\mathrm{d}x = \int_{1/3}^{2/3} 4\cdot(-3)\cdot 3\,\mathrm{d}x = -12 \\
K_{13} &= \int_0^1 c(x)\phi_0'(x)\phi_2'(x)\,\mathrm{d}x = 0
\end{aligned}$$

Since K is symmetric, the matrix is

$$K = \begin{pmatrix} 6 & -6 & 0 \\ -6 & 18 & -12 \\ 0 & -12 & 24 \end{pmatrix}$$

(d) (5 pts.) Construct the vector $\vec{F}$ for this problem. Show your work.

Solution: The vector $\vec{F}$ has matrix elements

$$\int_0^1 \delta(x - 2/3)\phi_i(x)\,\mathrm{d}x$$

Using the properties of the delta function, we find that

$$\begin{aligned}
F_1 &= \int_0^1 \delta(x-2/3)\phi_0(x)\,\mathrm{d}x = \phi_0(2/3) = 0 \\
F_2 &= \int_0^1 \delta(x-2/3)\phi_1(x)\,\mathrm{d}x = \phi_1(2/3) = 0 \\
F_3 &= \int_0^1 \delta(x-2/3)\phi_2(x)\,\mathrm{d}x = \phi_2(2/3) = 1
\end{aligned}$$

The vector $\vec{F}$ is thus

$$\vec{F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(e) **Bonus (5 pts.)**: Solve the equation $K\vec{U} = \vec{F}$ for $\vec{U}$ and write down the finite element solution $U(x)$. Graph the solution.

Solution: The equation is

$$\begin{pmatrix} 6 & -6 & 0 \\ -6 & 18 & -12 \\ 0 & -12 & 24 \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The first equation says that $6U_0 - 6U_1 = 0$, or $U_0 = U_1$. The second equation says $-U_0 + 3U_1 - 2U_2 = 0$, or $U_1 = U_2$ (since $U_0 = U_1$). The third equation says that $12U_2 = 1$, or $U_2 = 1/12$. We thus find that $U_0 = U_1 = U_2 = 1/12$, so the finite element solution is

$$U(x) = \frac{1}{12}\left(\phi_0(x) + \phi_1(x) + \phi_2(x)\right)$$

A plot of this solution is shown in Figure 2. It can be shown that the exact solution to the equation is

$$u(x) = \begin{cases} 1/12 & \text{if } 0 \leq x \leq 2/3 \\ \frac{1-x}{4} & \text{if } 2/3 < x \leq 1 \end{cases}$$

This is the same as the finite element solution $U(x)$.

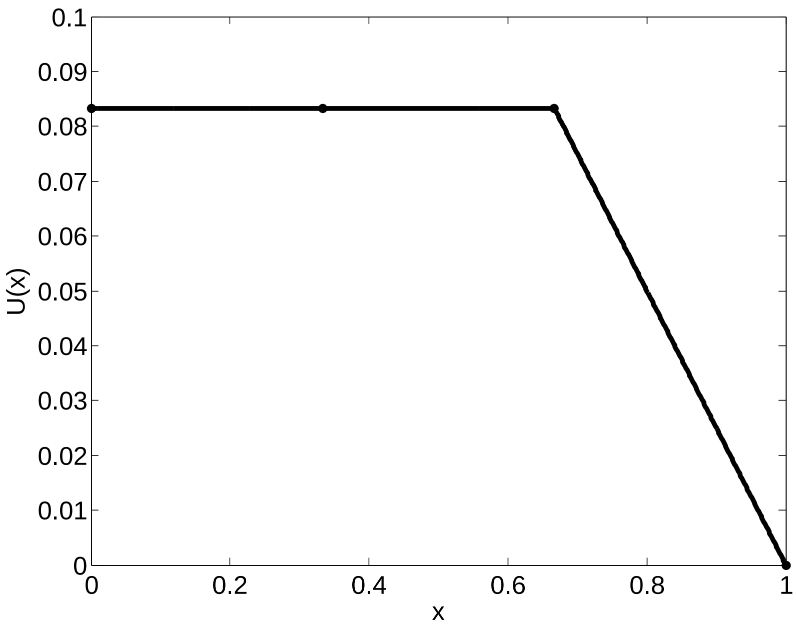


FIGURE 2. Solution to Question 5 (Bonus).