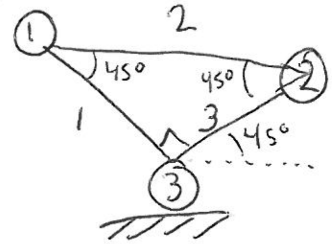


18.085 SUMMER 2013 - QUIZ 2 - JULY 26, 2013

YOUR NAME: Rosalie

(1) (5 × 6 = 30 points.)

Consider the truss drawn on the board.

a) Is there a possible mechanism or rigid motion? If so, draw one.

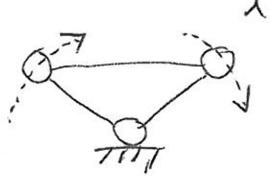
b) Build the matrix A corresponding to this truss.

c) Use the matrix A to verify mathematically your answer to (a). That is, depending on your answer to (a) : either prove the nullspace of A is trivial, or give a basis for the nullspace of A (no need to prove you have indeed a basis, but be sure of your answer!).

d) Now fix node 2. Get rid of bar 3 since it is now useless. What is the matrix A for this new truss?

e) Is there a mechanism or rigid motion for the truss in (d)? Prove this mathematically.

a) Yes! (rigid motion)



b)
$$A = \begin{array}{c|cccc} & u_1^H & u_1^V & u_2^H & u_2^V \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array} \quad \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

c) From (a), we expect $u_1^H = u_1^V = u_2^H = -u_2^V = \Delta$, or $\vec{u} = (\Delta \;\; \Delta \;\; \Delta \;\; -\Delta)^T$ to be in the nullspace of A, and indeed $A\vec{u} = \vec{0}$. Hence a basis for the null space is the given $\vec{u}$.

(The 3 rows of A are independent hence A has rank 3 hence the dimension of the nullspace is $4 - 3 = 1$, so we are done.)

d)
$$A = \begin{array}{c|cc} & u_1^H & u_1^V \\ \hline 1 & & \\ 2 & & \end{array} \quad \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 \end{pmatrix}$$

← upper-left block of old A! (so got rid of last 2 col^{ns}, last row)

e) No mechanism! No rigid motion! A is invertible (col^{ns} are independent, rows are too, $\det(A) = +1/\sqrt{2} \neq 0$), so the nullspace is trivial, the truss is stable.

(2) ($3 \times 8 + 2 \times 3 = 30$ points.)

Consider a hanging bar. We have the following equation for its displacement u, given that $c(x) = 1/2$ for $x < 1/2$ and $c(x) = 2$ for $x > 1/2$, and $f(x) = \cancel{1}$ ~~(constant force of 1 applied to every~~ part of the bar): $\delta(x - 1/2)$

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x), \qquad x \in [0, 1].$$

We also have the boundary conditions $u(0) = 0$ and $\frac{du}{dx}(1) = 0$.

a) Find $w(x) = c(x)\frac{du}{dx}$, and graph it.

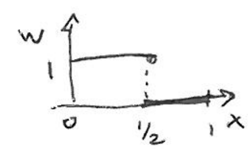
~~$w(x) = -\int_0^x 1\, ds = -x + C_1$, with $\frac{du}{dx}(1) = 0$ so $w(1) = 0$ too,~~

~~and $w(1) = -1 + C_1 = 0 \Rightarrow C_1 = 1 \Rightarrow w(x) =$~~

$$w(x) = -\int_0^x \delta(s - 1/2)\, ds = \begin{cases} 0 + C_1 & x < 1/2 \\ -1 + C_1 & \frac{1}{2} < x \end{cases}$$ and $\frac{du}{dx}(1) = 1$ so $w(1) = 0$

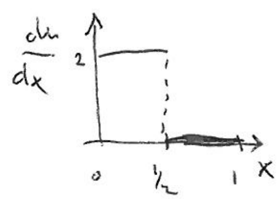
so $w(1) = 0 = -1 + C_1$

$$\Rightarrow C_1 = +1, \text{ and } w(x) = \begin{cases} 1 & x < 1/2 \\ 0 & 1/2 < x \end{cases}$$



b) Find $\frac{du}{dx}$, and graph it.

$$\frac{du}{dx} = \frac{w(x)}{c(x)} = \begin{cases} 2 & x < 1/2 \\ 0 & 1/2 < x \end{cases}$$

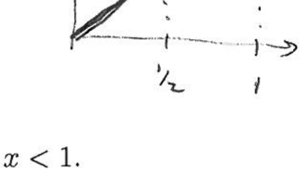


c) Find u, and graph it.

$$u = \int_0^x \frac{du}{dx}(s)\, ds = \begin{cases} \int_0^x 2\, ds = 2x + C_2 & x < 1/2 \\ \int_0^{1/2} 2\, ds + \int_{1/2}^x 0 = 1 + C_2 & \frac{1}{2} < x \end{cases}$$

and $u(0) = 0 = 2 \cdot 0 + C_2 \Rightarrow C_2 = 0$ so

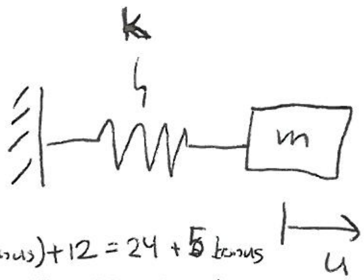
$$u(x) = \begin{cases} 2x & x < 1/2 \\ 1 & \frac{1}{2} < x \end{cases}$$



Circle your answers to (d), (e) (no explanation).

d) Which part of the bar is stiffer: $0 < x < 1/2$ or $1/2 < x < 1$.

e) Which part of the bar is stretched more: $0 < x < 1/2$ or $1/2 < x < 1$.



(5 bonus) + 12 = 24 + 5 bonus

(3) (5 + 5 + ~~10~~ + ~~10 = 30 points~~.)

We know that a system of springs and masses can be modeled using the $K = A^T CA$ framework we have seen in class, in particular, $M\vec{u}'' = -K\vec{u}$ for $\vec{u}$ the vector of displacements of the masses. We use primes to mean time derivatives. A is the first difference matrix and C is the constitutive law.

a) What does the equation $M\vec{u}'' = -K\vec{u}$ for $\vec{u}$ simplify to if we only have one mass and one spring? (Use the notation on the board.)

$mu'' = -ku$ (M, A, C are 1×1 matrices: $M = m$, $C = k$, $A = 1$)

b) Find an analytical solution to the equation $mu'' = -ku$, for $u = u(t)$. Don't bother with initial conditions. Just give ONE expression (there could be more than one that work, it could be complex) for u which satisfies the given equation.

$u(t) = e^{i\omega t}$, $\omega = \pm\sqrt{k/m}$ (guess $u = e^{i\omega t}$, then $u' = i\omega e^{i\omega t}$, $u'' = (i\omega)^2 e^{i\omega t} = -\omega^2 e^{i\omega t}$ so $u'' = -\omega^2 e^{i\omega t} = -(k/m)e^{i\omega t}$ $\Rightarrow \omega = \pm\sqrt{k/m}$). $u = Ae^{i\omega t}$, $= \sin \omega t$, $= \cos \omega t$ would work too....

BONUS! → c) Now find a solution of $mu'' = -ku$, for $u = u(t)$, with the following initial conditions: $u(0) = 1$ and $u'(0) = 0$.

We know $\cos 0 = 1$, $\frac{d}{dt}(\cos)\big|_0 = -\sin\big|_0 = 0$, so we guess $u(t) = \cos \omega t$, $\omega = \sqrt{k/m}$, and indeed it works:

$u' = (-\sin \omega t)\,\omega$, $u'' = -\omega^2 \cos \omega t = -k/m \cos \omega t$ so $\omega = \sqrt{k/m}$ as in (b), and $u(0) = \cos 0 = 1$, $u'(0) = -\omega \sin \omega t\big|_{t=0} = -\omega \cdot 0 = 0$

d) We want to solve the problem in (c) with $m = 1$ (so we will ignore m) and $k = 4$ using the leap-frog method (finite differences in time). Modify lines 4, 5 and 9 (don't modify anything else) of the following Matlab code to do this.

```
01: T=2*pi;nt=50;dt=T/nt;
02: k=4;
03: u=zeros(1,nt+1);
04: u0= 1;
05: up0= 0;
06: u(1)=u0;
07: u(2)=u0+dt*up0;
08: for i=3:nt+1
09:     u(i)=2*u(i-1) - u(i-2) +(dt^2)*(-k*u(i-1));
10: end
```

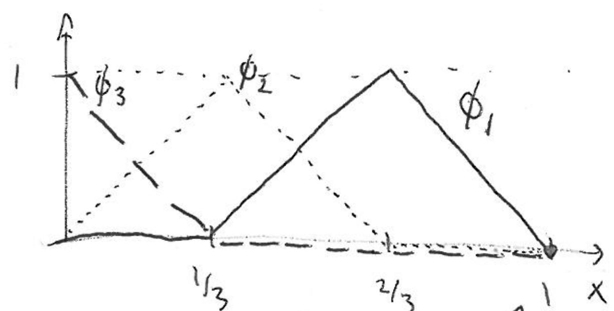
(4) (~~[illegible]~~ $6+10$ = ~~10~~ 16 points.)

(a) I want to use Newton's method to solve $p(x) = 0$ with starting guess x_0. Write down the algorithm to find the next guesses x_1, x_2, etc (1 line, plus say for which indices that line holds).

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{df}{dx}(x_n)} \quad , \; n = 0, 1, \ldots..$$

$$\left(\text{think} \quad u_{n+1} = u_n - \frac{g(u_n)}{\frac{dg}{du}(u_n)} \quad , \; n = 0, 1, \ldots \right)$$

(b) I want to solve Laplace's equation $-\frac{d^2u}{dx^2} = f$ on $x \in [0, 1]$ with boundary conditions (CAREFUL!) $u(1) = 0$ and $\frac{du}{dx}(0) = 0$. Let $h = 1/3$. Draw the hat functions you would use to solve this problem using the Finite Element Method. Label both your axes. Ok to draw them all on same graph.



(bdry cond^{ns} were flipped from what we had done in class, so flip your ϕ's!)

all ϕ's satisfy $\phi(1) = 0$
(b/c we want $u(1) = 0$,
or $U(1) = 0$, where
$U(x) = U_1\phi_1(x) + U_2\phi_2(x) + U_3\phi_3(x)$)