18.085: Summer 2015
Exam 1
July 10, 2015
Time Limit: 90 Minutes

Name (Print): SOLUTION

This exam contains 9 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use the textbook and any notes, but no electronic devices are allowed.

You are required to show your work on each problem on this exam, unless otherwise specified. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive very little credit; an incorrect answer supported by substantially correct calculations and explanations will still receive some partial credit.
- $\boxed{\text{Box}}$ **your final answer to each problem.**
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	30	
3	15	
4	20	
5	20	
Total:	100	

1. (15 points) Determine whether each of the statements is **True** or **False** and *explain why.*

(a) (3 points) A 4×5 matrix A can be full column rank.

$\boxed{\text{False}}$, the rank can be at most 4.

(b) (3 points) Consider a $m \times n$ matrix B. If $m > n$, the number of *independent* rows can be greater than the number of *independent* columns.

$\boxed{\text{False}}$, the number of independent rows always equals the number of independent columns.

(c) (3 points) If a matrix C is full column rank, then there is a unique solution to $C^\top C x = C^\top b$ for x.

$\boxed{\text{True}}$, If C is full column rank, then $C^T C$ is full rank (i.e. invertible).

Thus there is a unique solution $x = (C^T C)^{-1} C^T b$.

(d) (3 points) A singular (non-invertible) matrix D always has at least one eigenvalue equal to zero.

$\boxed{\text{True}}$, a non-invertible matrix has a null space, therefore there is some v such that $Dv = 0$. Thus v is an eigenvector with eigenvalue 0.

(e) (3 points) All orthogonal matrices Q preserve length, i.e. $|Qx| = |x|$ for all x.
(Recall: $Q^\top Q = I$ and $|x|^2 = x^\top x$.)

$\boxed{\text{True}}$,

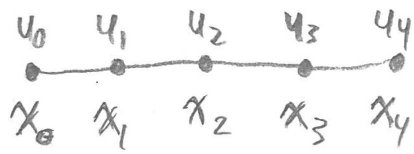
$$|Qx| = \sqrt{(Qx)^T (Qx)} = \sqrt{x^T \underbrace{Q^T Q}_{I} x} = \sqrt{x^T x} = |x|.$$

2. (30 points) Consider the following boundary value problem:

$$\frac{d^2u}{dx^2} = f(x)$$

with $f(x) = 8 - 2\delta(x - \frac{1}{2})$ and boundary conditions $u'(0) = 0$ and $u(1) = 0$.

(a) (15 points) Setup *but do not solve* the discretized problem in matrix form $Au = b$ with a grid spacing of $h = 1/4$. The solution vector u to this linear system is our approximation to $u(x)$ at the grid points.



Node 0: $\dfrac{u_1 - u_0}{h} = 0$

$h = \frac{1}{4}$

$h^2 = \frac{1}{16}$

Node 1: $\dfrac{u_2 - 2u_1 + u_0}{h^2} = 8$

Node 2: $\dfrac{u_3 - 2u_2 + u_1}{h^2} = 8 - \dfrac{2}{h} = 8 - \dfrac{2}{(1/4)} = 0$

Node 3: $\dfrac{u_4 - 2u_3 + u_2}{h^2} = 8$

Node 4: $u_4 = 0$

$$\boxed{\begin{bmatrix} -4 & 4 & 0 & 0 & 0 \\ 16 & -32 & 16 & 0 & 0 \\ 0 & 16 & -32 & 16 & 0 \\ 0 & 0 & 16 & -32 & 16 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 8 \\ 0 \end{bmatrix}}$$

(b) (15 points) Find the exact solution for $u(x)$. Sketch the result.

(Suggestion: first write down the general form of the solution on either side of $x = \frac{1}{2}$. Then apply the boundary conditions, jump condition of $\frac{du}{dx}$ at $x = \frac{1}{2}$, and continuity condition of $u(x)$ at $x = \frac{1}{2}$.).

Away from $x=\frac{1}{2}$,

$$\frac{d^2u}{dx^2} = 8$$

$$\frac{du}{dx} = 8x + c$$

$$u = 4x^2 + cx + d$$

$$u(x) = \begin{cases} 4x^2 + cx + d & 0 \le x < \frac{1}{2} \\ 4x^2 + e(x-1) + f & \frac{1}{2} < x \le 1 \end{cases}$$

$$\frac{du}{dx}(x) = \begin{cases} 8x + c & 0 \le x < \frac{1}{2} \\ 8x + e & \frac{1}{2} < x \le 1 \end{cases}$$

Boundary conditions: $u'(0) = 0 \rightarrow u'(0) = 8(0) + c = 0 \rightarrow \underline{c = 0}$

$$u(1) = 0 \rightarrow u(1) = 4(1)^2 + e(1-1) + f = 0 \rightarrow \underline{f = -4}$$

Jump condition at $x=\frac{1}{2}$: $\left[\frac{du}{dx}\right]\left(\frac{1}{2}\right) = -2$

$$\left[8\left(\tfrac{1}{2}\right) + \underbrace{\cancel{c}}_{0}\right] - \left[8\left(\tfrac{1}{2}\right) + e\right] = -2$$

$$-4 - e = -2$$

$$\underline{e = -2}$$

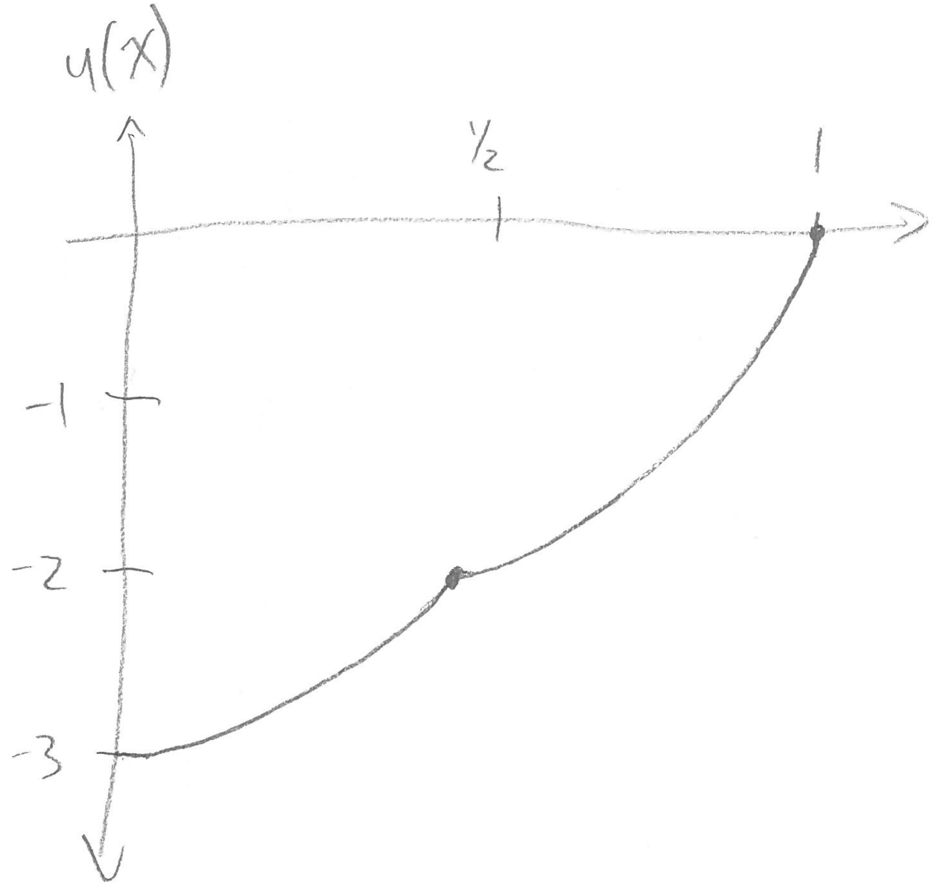
Continuity condition at $x=\frac{1}{2}$: $4\left(\frac{1}{2}\right)^2 + \underbrace{\cancel{c}}_{0}\left(\frac{1}{2}\right) + d = 4\left(\frac{1}{2}\right)^2 + e\left(\frac{1}{2} - 1\right) + f$

$$\cancel{1} + d = \cancel{1} - 2\left(-\tfrac{1}{2}\right) - 4$$

$$d = 1 - 4 \longrightarrow \underline{d = -3}$$

$$\boxed{u(x) = \begin{cases} 4x^2 - 3 & 0 \le x \le \frac{1}{2} \\ 4x^2 - 2(x-1) - 4 & \frac{1}{2} \le x \le 1 \end{cases}}$$

(b)



3. (15 points) We are given the four points (0,0), (1,0), (2,1), and (3,5) to which we want to fit a curve of the form $y = ax^2$. Setup *and solve* the normal equations, the solution to which gives us the least-squares estimate for a. Sketch the data points and the best-fit curve.

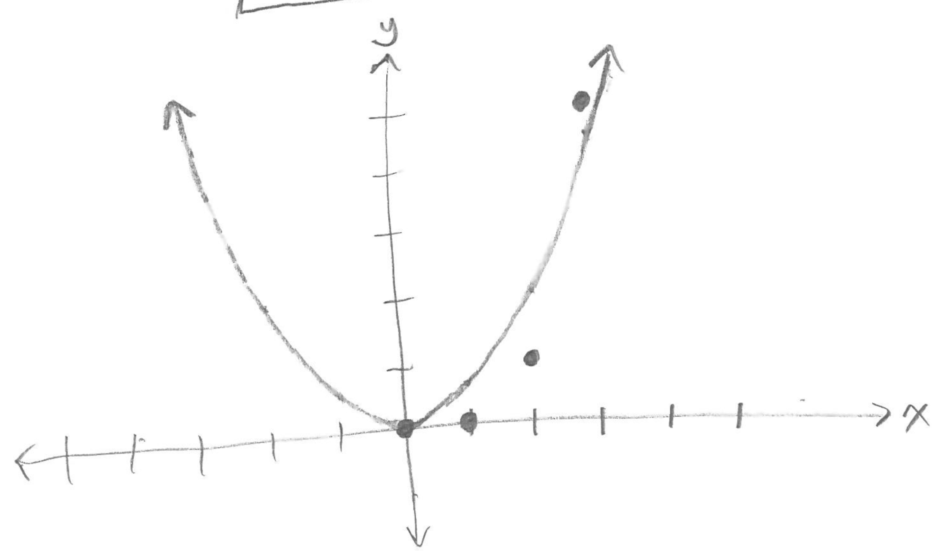
$$A^T A \hat{u} = A^T b$$

$$\begin{bmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \end{bmatrix} \begin{bmatrix} a \end{bmatrix} = \begin{bmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} a = \begin{bmatrix} 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$$

$$98a = 49$$

$$\boxed{a = 1/2}$$



4. (20 points) Consider the matrix $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

(a) (7 points) Find the eigenvalues and eigenvectors of A.

$$\det\left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \lambda I\right) = \det \begin{bmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 4 = 0$$

$$9 - 6\lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 - 6\lambda + 5 = 0$$
$$(\lambda - 5)(\lambda - 1) = 0$$
$$\lambda_1 = 5$$
$$\lambda_2 = 1$$

$$\lambda_1 = 5 \quad N\left(\begin{bmatrix} 3-5 & 2 \\ 2 & 3-5 \end{bmatrix}\right) = N\left(\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\right) \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \quad N\left(\begin{bmatrix} 3-1 & 2 \\ 2 & 3-1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right) \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\boxed{\text{A has eigenvalue 5 with eigenvector } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and eigenvalue 1 with eigenvector } \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

(b) (2 points) Is A positive definite? Explain why or why not.

$\boxed{\text{Yes}}$: A is symmetric, and all eigenvalues are positive.

(c) (3 points) What is the value of $A^{99}v$ when $v = \begin{bmatrix} -99 \\ 99 \end{bmatrix}$?

(Suggestion: Use the results of part (a)).

Note: v is in direction of x_2: $v = -99x_2$

$$A^{99}v = A^{99}(-99x_2) = -99A^{99}x_2 = -99\lambda_2^{99}x_2 = -99x_2$$

$$= \boxed{\begin{bmatrix} -99 \\ 99 \end{bmatrix} = v}$$

(d) (6 points) Find the singular value decomposition of A.

A is symmetric positive definite, so we can write

$$A = U \Sigma U^T$$

$$u_1 = \frac{x_1}{|x_1|} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad u_2 = \frac{x_2}{|x_2|} = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$$

(e) (2 points) What is the condition number of A?

For a positive definite matrix,

$$c(A) = \frac{\lambda_{max}}{\lambda_{min}}$$

$$= \boxed{5}$$

5. (20 points) Consider the following initial value problem for $x(t)$:

$$\frac{dx}{dt} = -\left(\frac{\lambda}{t+1}\right)x$$

with initial condition $x(0) = x_0 > 0$ and $\lambda > 0$ is a constant.

(a) (4 points) Find the exact solution for $x(t)$.
(Suggestion: Separate variables and integrate both sides).

$$\int \frac{dx}{x} = -\lambda \int \frac{dt}{t+1}$$

$$\ln(x) = -\lambda \ln(t+1) + c$$

$$\ln(x) = \ln\left(\frac{1}{(t+1)^\lambda}\right) + c$$

$$x = \frac{k}{(t+1)^\lambda} \xrightarrow[\substack{\Downarrow \\ k = x_0}]{x(0) = x_0} \boxed{x(t) = \frac{x_0}{(t+1)^\lambda}}$$

(b) (8 points) We would like to solve this initial value problem numerically with a time step Δt. Recall that $x_n = x(n\Delta t)$. Derive an *explicit* formula for x_{n+1} as a function of x_n, n, Δt, and λ using a Forward Euler method.

$$\frac{x_{n+1} - x_n}{\Delta t} = \frac{-\lambda}{n\Delta t + 1} x_n$$

$$\boxed{x_{n+1} = \left(1 - \frac{\lambda \Delta t}{n\Delta t + 1}\right) x_n}$$

(c) (8 points) Derive an *explicit* formula for x_{n+1} as a function of x_n, n, Δt, and λ using a Trapezoidal Rule method.

$$\frac{x_{n+1}-x_n}{\Delta t} = \left(\frac{\frac{-\lambda}{n\Delta t+1}x_n + \frac{-\lambda}{(n+1)\Delta t+1}x_{n+1}}{2}\right)$$

$$\frac{x_{n+1}-x_n}{\Delta t} = \frac{-\lambda}{2n\Delta t+2}x_n + \frac{-\lambda}{2(n+1)\Delta t+2}x_{n+1}$$

$$x_{n+1}-x_n = \frac{-\lambda\Delta t}{2n\Delta t+2}x_n + \frac{-\lambda\Delta t}{2(n+1)\Delta t+2}x_{n+1}$$

$$\boxed{x_{n+1} = \frac{\left(1-\frac{\lambda\Delta t}{2n\Delta t+2}\right)}{\left(1+\frac{\lambda\Delta t}{2(n+1)\Delta t+2}\right)}x_n}$$