

18.085: Summer 2015  
Exam 1  
July 10, 2015  
Time Limit: 90 Minutes

Name (Print): \_\_\_\_\_

This exam contains 9 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use the textbook and any notes, but no electronic devices are allowed.

You are required to show your work on each problem on this exam, unless otherwise specified. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive very little credit; an incorrect answer supported by substantially correct calculations and explanations will still receive some partial credit.
- **Box** your final answer to each problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	30	
3	15	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

1. (15 points) Determine whether each of the statements is **True** or **False** and *explain why*.
  - (a) (3 points) A  $4 \times 5$  matrix  $A$  can be full column rank.
  - (b) (3 points) Consider a  $m \times n$  matrix  $B$ . If  $m > n$ , the number of *independent* rows can be greater than the number of *independent* columns.
  - (c) (3 points) If a matrix  $C$  is full column rank, then there is a unique solution to  $C^\top Cx = C^\top b$  for  $x$ .
  - (d) (3 points) A singular (non-invertible) matrix  $D$  always has at least one eigenvalue equal to zero.
  - (e) (3 points) All orthogonal matrices  $Q$  preserve length, i.e.  $|Qx| = |x|$  for all  $x$ .  
(Recall:  $Q^\top Q = I$  and  $|x|^2 = x^\top x$ .)

2. (30 points) Consider the following boundary value problem:

$$\frac{d^2u}{dx^2} = f(x)$$

with  $f(x) = 8 - 2\delta(x - \frac{1}{2})$  and boundary conditions  $u'(0) = 0$  and  $u(1) = 0$ .

- (a) (15 points) Setup *but do not solve* the discretized problem in matrix form  $Au = b$  with a grid spacing of  $h = 1/4$ . The solution vector  $u$  to this linear system is our approximation to  $u(x)$  at the grid points.

(b) (15 points) Find the exact solution for  $u(x)$ . Sketch the result.

(Suggestion: first write down the general form of the solution on either side of  $x = \frac{1}{2}$ . Then apply the boundary conditions, jump condition of  $\frac{du}{dx}$  at  $x = \frac{1}{2}$ , and continuity condition of  $u(x)$  at  $x = \frac{1}{2}$ .)

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3. (15 points) We are given the four points  $(0,0)$ ,  $(1,0)$ ,  $(2,1)$ , and  $(3,5)$  to which we want to fit a curve of the form  $y = ax^2$ . Setup *and solve* the normal equations, the solution to which gives us the least-squares estimate for  $a$ . Sketch the data points and the best-fit curve.

4. (20 points) Consider the matrix  $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ .

(a) (7 points) Find the eigenvalues and eigenvectors of  $A$ .

(b) (2 points) Is  $A$  positive definite? Explain why or why not.

(c) (3 points) What is the value of  $A^{99}v$  when  $v = \begin{bmatrix} -99 \\ 99 \end{bmatrix}$ ?

(Suggestion: Use the results of part (a)).

(d) (6 points) Find the singular value decomposition of  $A$ .

(e) (2 points) What is the condition number of  $A$ ?

5. (20 points) Consider the following initial value problem for  $x(t)$ :

$$\frac{dx}{dt} = - \left( \frac{\lambda}{t+1} \right) x$$

with initial condition  $x(0) = x_0 > 0$  and  $\lambda > 0$  is a constant.

- (a) (4 points) Find the exact solution for  $x(t)$ .

(Suggestion: Separate variables and integrate both sides).

- (b) (8 points) We would like to solve this initial value problem numerically with a time step  $\Delta t$ . Recall that  $x_n = x(n\Delta t)$ . Derive an *explicit* formula for  $x_{n+1}$  as a function of  $x_n$ ,  $n$ ,  $\Delta t$ , and  $\lambda$  using a Forward Euler method.



- (c) (8 points) Derive an *explicit* formula for  $x_{n+1}$  as a function of  $x_n$ ,  $n$ ,  $\Delta t$ , and  $\lambda$  using a Trapezoidal Rule method.