Problem Set 4 Solutions

1) a) A_1 is positive definite. It is triangular, so eigenvalues are along the diagonal. $\Rightarrow$ $\lambda_i > 0$.

A_2 is not positive definite. It is not full rank. (col ③ = col ①)

A_3 is not positive definite. The upper left element < 0, so it does not pass the upper left determinant test.

b) If a matrix B is not full rank, then it has a null space, which means $Bv = 0$ for some non-zero v. This corresponds to a zero-eigenvalue which is not > 0.

c) Since C is symmetric it has a complete, orthonormal eigenbasis. So we can write any vector u as a combination of these eigenvectors.

$$u = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \quad \text{where } Cv_i = \lambda_i v_i$$

Substituting into

$$u^T C u = (c_1 v_1 + c_2 v_2 + \dots + c_n v_n)^T C (c_1 v_1 + c_2 v_2 + \dots c_n v_n)$$

$$= (c_1 v_1 + c_2 v_2 + \dots + c_n v_n)^T (c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_n \lambda_n v_n)$$

Note:

$$\left.\begin{aligned} v_i^T v_j &= 1 \text{ if } i = j \\ &= 0 \text{ if } i \neq j \end{aligned}\right\} \rightarrow \quad = c_1^2 \lambda_1 + c_2^2 \lambda_2 + \dots + c_2^2 \lambda_n$$

$$> 0 \text{ if } \lambda_i > 0$$

d) $$u^T(D_1+D_2)u = \underbrace{u^T D_1 u}_{\substack{>0 \\ \text{since } D_1 \text{ is} \\ \text{positive definite}}} + \underbrace{u^T D_2 u}_{\substack{>0 \\ \text{since } D_2 \text{ is} \\ \text{positive definite}}} > 0$$

Thus the matrix (D_1+D_2) is also positive definite.

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② (a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

eigenvalues: of A^TA

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-3)(\lambda-1) = 0$$

$$\lambda = 3, 1$$

$\lambda_1 = 3 = \sigma_1^2 \Rightarrow \boxed{\sigma_1 = \sqrt{3}}$

$\lambda_2 = 1 = \sigma_2^2 \Rightarrow \boxed{\sigma_2 = 1}$

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v_1: $\lambda_1 = 3 \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $N(A^TA - \lambda_1 I) = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$ $|v_1| = 1$

v_2: $\lambda_2 = 1 \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $N(A^TA - \lambda_2 I) = c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\boxed{v_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}}$ $|v_2| = 1$

$$u_1 = \frac{A v_1}{\sigma_1} = \frac{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 2/\sqrt{2} \\ 0 \end{bmatrix}}{\sqrt{3}} = \boxed{\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \end{bmatrix} = u_1}$$

$$u_2 = \frac{A v_2}{\sigma_2} = \frac{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}}{1} = \boxed{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = u_2}$$

u_3 + u_4 correspond to any two orthonormal eigenvectors of AA^T, with eigenvalue 0. (i.e. vectors in $N(AA^T)$).

$$AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice: col 3 = col 1 + col 2

col 4 = 0

$$u_3 = \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}}{\sqrt{1^2+1^2+(-1)^2}} = \boxed{\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{bmatrix} = u_3}$$

$$\boxed{u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$A = U \Sigma V^T$$

$$A = \begin{array}{c} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} u_1 & \quad u_2 & \quad u_3 & \quad u_4 \end{matrix} \end{array} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \end{array} \begin{array}{c} v_1^T \\ v_2^T \end{array}$$

(b) Note $B = A^T = (U \Sigma V^T)^T = V \Sigma^T U^T$

Note: if Q is orthogonal, so is Q^T

$$B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) $C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$C^T C = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$

eigenvalues of $C^T C$:

$$\begin{vmatrix} 10-\lambda & 20 \\ 20 & 40-\lambda \end{vmatrix} = 0$$

$$(10-\lambda)(40-\lambda) - 20^2 = 0$$

$$\cancel{400} - 50\lambda + \lambda^2 - \cancel{400} = 0$$

$$\lambda(\lambda - 50) = 0$$

$\lambda_1 = 50$, $\lambda_2 = 0$ $\rightarrow$ zero eigenvalue since $C^T C$ has a null space.

$$\Downarrow \qquad \Downarrow$$

$$\boxed{\sigma_1 = 5\sqrt{2}, \quad \sigma_2 = 0}$$

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$v_1 : \lambda_1 = 50 \qquad \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} - \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -40 & 20 \\ 20 & -10 \end{bmatrix} \qquad N(C^TC - \lambda_1 I) = c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\boxed{v_1 = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\sqrt{1^2+2^2}} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}}$$

$v_2 : \lambda_2 = 0 \qquad N(C^TC - \lambda_2 I) = N(A) = c\begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \boxed{v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}}$

$$u_1 = \frac{C v_1}{\sigma_1} = \frac{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}}{5\sqrt{2}} = \frac{\begin{bmatrix} 5/\sqrt{5} \\ 15/\sqrt{5} \end{bmatrix}}{5\sqrt{2}} \Rightarrow \boxed{u_1 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}}$$

u_2 : unit vector $\perp$ to u_1 $\Rightarrow$ $\boxed{u_2 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}}$

$$\boxed{A = U \Sigma V^T = \begin{array}{c} \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \\ \begin{array}{cc} u_1 & \quad u_2 \end{array} \end{array} \begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{array}{cc} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} & \begin{array}{c} v_1^T \\ v_2^T \end{array} \end{array}}$$

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>> A=[1.2969 0.8648; 0.2161 0.1441];
>> b1=[1.2969;0.2161];
>> xa = A \ b1

xa =

     1
     0

>> b2 = round(10^3*b1)/10^3;
>> xb = A \ b2

xb =

   1.0e+04 *

    1.0090
   -1.5130

>> B = round(10^3*A)/10^3;
>> xc = B \ b1

xc =

    2.4014
   -2.1014

>> c = cond(A)

c =

   2.4973e+08

>>
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→ The answers differ so much because the condition number of A is very large.

(4) $c(A) = \kappa = \dfrac{\sigma_{max}}{\sigma_{min}}$

(a) condition number of A^TA:

find eigenvalues of $(A^TA)^T A^TA = (A^TA)(A^TA)$

symmetric

Notice: eigenvectors of A^TA will be eigenvectors of $(A^TA)^2$

$$(A^TA)(A^TA)v_i = (A^TA)\sigma_i^2 v_i = \sigma_i^2 (A^TA) v_i = \sigma_i^4 v_i$$

So singular values of A^TA are squares of singular values of A.

$$\Rightarrow \boxed{c(A^TA) = \frac{\sigma_{max}^2}{\sigma_{min}^2} = \kappa^2}$$

(b) condition number of A^{-1}:

find eigenvalues of $(A^{-1})^T A^{-1} = (A^{T^{-1}} A^{-1}) = (AA^T)^{-1}$

Notice: eigenvectors of AA^T will be eigenvectors of $(AA^T)^{-1}$

$$AA^T v_i = \sigma_i^2 v_i$$

$$\Downarrow$$

$$(AA^T)^{-1} v_i = \frac{1}{\sigma_i^2} v_i$$

So singular values of A^{-1} are inverses of singular values of A.

$$\Rightarrow \boxed{c(A^{-1}) = \frac{1/\sigma_{min}}{1/\sigma_{max}} = \frac{\sigma_{max}}{\sigma_{min}} = \kappa}$$

(c) If A is orthogonal then $A^TA = I$ by definition.

Thus, A^TA has all eigenvalues $= 1 \Rightarrow \sigma_i^2 = 1 \Rightarrow \sigma_i = 1$ for $i = 1..n$

$$\Rightarrow \boxed{\text{cond}(Q) = 1}$$

(d) The smallest condition number occurs when $\sigma_{max} = \sigma_{min}$,

thus the minimum condition number is 1.

Example, any 3x3 orthogonal matrix:

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

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5 (a) $$\frac{dx}{dt} = -x^2 \qquad x(0) = 1$$

$$-\frac{dx}{x^2} = dt$$

$$\frac{1}{x} = t + c$$

$$x = \frac{1}{t+c} \overset{x(0)=1}{\Longrightarrow} 1 = \frac{1}{c} \Longrightarrow c = 1$$

$$\boxed{x(t) = \frac{1}{t+1}} \qquad \text{At } t=1, \quad \boxed{x(1) = \frac{1}{2}}$$

(b) $$\frac{x_{n+1} - x_n}{\Delta t} = -x_n^2$$

$$\boxed{x_{n+1} = x_n - \Delta t \, x_n^2}$$

(c) See attached.

(d) The forward Euler method is 1st order in Δt so if we reduce Δt by a factor of 10, we expect the error to decrease by approximately 10.

(e) See attached. The error is reduced by approximately a factor of 10, confirming our prediction in (d).

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(f) (i) $$\frac{x_{n+1} - x_n}{\Delta t} = -x_{n+1}^2$$

$$\Delta t\, x_{n+1}^2 + x_{n+1} - x_n = 0$$

$$x_{n+1} = \frac{-1 \pm \sqrt{1 + 4 x_n \Delta t}}{2\Delta t}$$

choose "+" solution so that $x_{n+1} > 0$ if $x_n > 0$.

$$\boxed{x_{n+1} = \frac{-1 + \sqrt{1 + 4 x_n \Delta t}}{2\Delta t}}$$

(ii) $$\frac{x_{n+1} - x_n}{\Delta t} = \frac{-x_{n+1}^2 - x_n^2}{2}$$

$$\Delta t\, x_{n+1}^2 + 2x_{n+1} + \left(\Delta t\, x_n^2 - 2x_n\right) = 0$$

$$x_{n+1} = \frac{-2 \pm \sqrt{4 - 4\Delta t\left(\Delta t\, x_n^2 - 2x_n\right)}}{2\Delta t}$$

choose "+" soln. again for same reasoning.

$$\boxed{x_{n+1} = \frac{-1 \pm \sqrt{1 - \Delta t\left(\Delta t\, x_n^2 - 2x_n\right)}}{\Delta t}}$$

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```
function x1=ivp(dt)
% x1 = approximate value of x(t=1)
% dt = timestep
steps=1/dt; % number of total timesteps
x=1; % initial condition
for i=1:steps;
    x=x-dt*x^2;
end
x1=x;
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>> x1c=ivp(0.1)

x1c =

    0.4817

>> e1c=0.5-x1c

e1c =

    0.0183

>> x1e=ivp(0.01)

x1e =

    0.4983

>> e1e=0.5-x1e

e1e =

    0.0017

>> e1c/e1e

ans =

   10.4987

>>
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