Problem Set 4

Unless otherwise specified, you may use MATLAB to assist with computations. Please provide a print-out of the code used and its output with your assignment.

1. Positive definiteness.

   (a) Without doing any calculations, determine whether the following matrices are positive definite (and explain your reasoning):

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} -1 & 2 & 4 \\ 2 & 8 & -9 \\ 4 & -9 & 3 \end{bmatrix}.$$

   (b) Show that if a $n \times n$ symmetric matrix B is not full rank, it cannot be positive definite.

   (c) If all eigenvalues of a $n \times n$ symmetric matrix C are positive, show that $u^\top C u > 0$ for every $u \neq 0$.

   (d) Show that if the symmetric $n \times n$ matrices D_1 and D_2 are positive definite, then so is $D_1 + D_2$.

2. SVD. **This problem is to be done by hand.** Find the singular value decomposition of the following matrices:

   (a)
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix},$$

   (b)
$$B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$
   (Hint: Use the result from part (a))

   (c)
$$C = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

3. Error. Consider the matrix:

$$A = \begin{bmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{bmatrix}.$$

   (a) Set `b1 = [1.2969; 0.2161]` and compute `x = A \ b1` in MATLAB.

   (b) Repeat the process but with a vector `b2` obtained from `b1` by rounding each value to three decimal places. Why do the two answers differ by so much?

   (c) Now compute `x = B \ b1` where B is obtained from A by rounding each value to three decimal places. As you will see, our solution is also sensitive to small changes in the matrix A.

4. Condition number. If the condition number of an invertible $n \times n$ matrix A is κ, find the condition number of the following matrices in terms of κ:

   (a) $A^\top A$

   (b) A^{-1}

   (c) What is the value of κ if A is an orthogonal matrix?

   (d) What is the smallest possible value for κ? Construct a 3×3 matrix with this minimum condition number.

5. Initial Value Problem. (**Not to be handed in**). Consider the following initial value problem for $x(t)$:

$$\frac{dx}{dt} = -x^2$$

   with initial condition $x(0) = 1$.

   (a) Find the exact solution for $x(t)$. (Hint: Separate variables and integrate both sides). Compute the exact value of $x(1)$.

   (b) We would like to setup this problem to be solved numerically with a time step Δt. Recall that $x_n = x(n\Delta t)$. Find an explicit formula for x_{n+1} as a function of x_n, Δt, and λ using a Forward Euler method.

   (c) Implement this explicit numerical method in MATLAB and find the approximation of $x(1)$ using $\Delta t = \frac{1}{10}$. Calculate the difference between the exact value and this approximate value.

   (d) If we reduce the time step from $\Delta t = \frac{1}{10}$ to $\Delta t = \frac{1}{100}$, how much you expect the error to be reduced?

   (e) Find the approximation of $x(1)$ with $\Delta t = \frac{1}{100}$. Calculate the difference between the exact value and this approximate value. By what factor is the error at $x(1)$ reduced by switching from $\Delta t = \frac{1}{10}$ to $\Delta t = \frac{1}{100}$?

   (f) Derive an explicit formula for x_{n+1} as a function of x_n, Δt, and λ using a:

      i. Backward Euler method,

      ii. Trapezoidal Rule method.