AY

Problem Set 3 Solutions

1) u satisfies $Au = b$

Normal equations: $A^T A \hat{u} = A^T b$

take $\hat{u} = u \longrightarrow A^T A u = A^T (Au)$

$A^T A u = A^T A u$ ✓

Thus $\hat{u} = u$ is a solution to the normal equations.

$$A^T A \hat{u} = A^T b$$

will have a unique solution for $\hat{u}$

if $A^T A$ is full rank $\Rightarrow$ A must be full column rank.

(invertible)

②.

② a) Need two linearly independent vectors on the plane $x+y+z=0$

$\rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

If $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ were on the plane, then we could solve $\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Projection of b onto plane defined by columns of $A \Rightarrow A\hat{u}$

Solve normal equations for $\hat{u}$ $\Rightarrow$ In MATLAB $A \backslash b = \hat{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$p = A\hat{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\Rightarrow$ nearest point is $\boxed{\begin{aligned} x_1 &= -1 \\ x_2 &= 0 \\ x_3 &= 1 \end{aligned}}$

b) The line $\{x = 2y, y = 2z\}$ is defined by the vector $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Want to project $\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ onto $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ and see which has smaller $|e| = |b - A\hat{u}|$

<u>Point a</u>

$\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} u = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \longrightarrow A^T A \hat{u} = A^T b$

$\begin{bmatrix} 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \hat{u} = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

$21 \hat{u} = 21$

$\hat{u} = 1$

$e_a = b - A\hat{u}$

$e_a = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} 1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

$|e_a| = \sqrt{1^2 + (-2)^2 + 0^2}$

$|e_a| = \sqrt{5}$

3.

Point b

$$\begin{bmatrix}4\\2\\1\end{bmatrix} u = \begin{bmatrix}3\\3\\3\end{bmatrix} \rightarrow A^T A \hat{u} = A^T b$$

$$\begin{bmatrix}4 & 2 & 1\end{bmatrix}\begin{bmatrix}4\\2\\1\end{bmatrix}\hat{u} = \begin{bmatrix}4 & 2 & 1\end{bmatrix}\begin{bmatrix}3\\3\\3\end{bmatrix}$$

$$21\ \hat{u} = 21$$

$$\hat{u} = 1$$

$$e_b = b - A\hat{u}$$

$$= \begin{bmatrix}3\\3\\3\end{bmatrix} - \begin{bmatrix}4\\2\\1\end{bmatrix} 1 = \begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$$|e_b| = \sqrt{(-1)^2 + 1^2 + 2^2}$$

$$|e_b| = \sqrt{6}$$

$\boxed{|e_a| < |e_b|, \text{ so point } a \text{ is closer to the line.}}$

(3) $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

a) $Au = b \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow[\text{elimination}]{} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{array}\right] \Rightarrow 0 = 4 \Rightarrow$ no solution!

b) $A^T A \hat{u} = A^T b$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \hat{u} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \hat{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{matrix} \rightarrow u_1 = 4/3 \\ \rightarrow u_2 = 1 \end{matrix} \qquad \boxed{\hat{u} = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}}$$

c) $A = QR$

q_1: $\dfrac{a_1}{|a_1|} = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{1^2+1^2+1^2}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = q_1$

q_2: $a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 0 q_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow[\text{normalize}]{} q_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \qquad R = Q^T A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

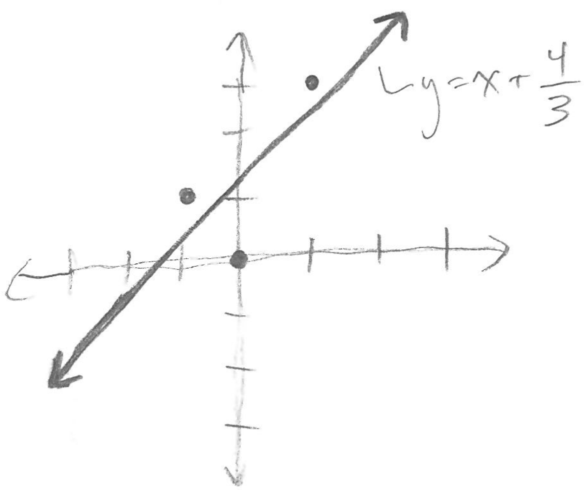
5).

$$R\hat{u} = Q^T b$$

$$\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \hat{u} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{3} \\ 2/\sqrt{2} \end{bmatrix} \begin{matrix} \to u_1 = 4/3 \\ \to u_2 = 1 \end{matrix}$$

$$\boxed{\hat{u} = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}}$$

d) This corresponds to fitting the points $(-1,1)$, $(0,0)$, $(1,3)$ to a line. The best line is $y = x + \frac{4}{3}$.



4) a) $H = I - 2uu^T$

i) • $H^T = (I - 2uu^T)^T$

$= I^T - (2uu^T)^T$

$= I - 2u^{T^T}u^T = I - 2uu^T = H \rightarrow$ symmetric

• $H^T H = (I - 2uu^T)^T (I - 2uu^T)$

$= (I - 2uu^T)(I - 2uu^T)$

$= I^2 - 4uu^T I + 4u\underbrace{u^T u}_{=1 \text{ since } u = \text{unit vector}} u^T$

$= I - 4uu^T + 4uu^T = I \rightarrow$ orthogonal

• Since $H = H^T$ and $H^T H = I$,

$HH = I \Rightarrow H = H^{-1}$

• H is invertible, so it is full rank.

ii) $Hv = \lambda v$, H is symmetric so we have orthogonal eigenvectors.

if $v = u \Rightarrow \lambda = -1$ (flips direction)

if $v \perp u \Rightarrow \lambda = 1$ (unchanged)

$$\boxed{\text{So we have } \lambda_1 = -1 \text{ and } \lambda_i = 1 \text{ for } i = 2 \dots n}$$

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b) $D = I - uu^T$

i) Consider a vector $v = w + cu$ ← portion parallel to u
portion $\perp$ to u

$$D(w+cu) = (I - uu^T)(w+cu)$$
$$= (I-uu^T)w + (I-uu^T)cu$$
$$= w - u\underbrace{u^Tw}_{\substack{=0 \\ \text{since } u \perp w}} + cu - cu\underbrace{u^Tu}_{\substack{=1 \\ \text{since } u \text{ is unit vector}}}$$

$$= w + cu - cu = w$$

- So D eliminates component of v inline with u.
- Further applications of D have no additional effect! (component inline with u already removed).

$\Rightarrow D^{42}$ is the <u>same</u> transformation as D.

ii) D is <u>not</u> full rank as $Du = 0 \rightarrow u \in N(D)$.

The vector u forms an orthonormal basis for $N(D)$, as $\dim(N(D)) = 1$.

iii) D has one zero eigenvalue corresponding to the eigenvector u.

$$Du = \lambda u = 0 \Rightarrow \lambda = 0.$$

Note that $\dim(N(D)) = \#$ of zero eigenvalues.

8.

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$$\frac{d}{dt}\begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} r \\ w \end{bmatrix}$$

eigenvalues of A: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 6-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(1-\lambda) + 4 = 0$$

$$6 - 7\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 7\lambda + 10 = (\lambda-5)(\lambda-2) = 0$$

$$\lambda = 2, 5$$

$\lambda_1 = 2$: $A - \lambda_1 I = \begin{bmatrix} 6-2 & -2 \\ 2 & 1-2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \longrightarrow N(A-\lambda_1 I) = c\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = 5$: $A - \lambda_1 I = \begin{bmatrix} 6-5 & -2 \\ 2 & 1-5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \rightarrow N(A - \lambda_2 I) = c\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

At $t=0$: $\begin{bmatrix} r(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \end{bmatrix} = 10\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 10\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 10v_1 + 10v_2$

Solution $\Rightarrow$ $\boxed{\begin{bmatrix} r(t) \\ w(t) \end{bmatrix} = 10e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \underbrace{10e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}}$

this term dominates for $t \to \infty$,

so $r/w = 2$ as $t \to \infty$.