Problem Set 3

Unless otherwise specified, you may use MATLAB to assist with computations. Please provide a print-out of the code used and its output with your assignment.

1. Suppose that $Au = b$ has a solution u. Show that $\hat{u} = u$ is a solution to the normal equations. What must be true about A for $\hat{u}$ to be unique?

2. Projections.

   (a) Find the point on the plane $x+y+z = 0$ nearest to the point $(x = 1, y = 2, z = 3)$.

   (b) Point a has coordinates $(x = 5, y = 0, z = 1)$ and point b has coordinates $(x = 3, y = 3, z = 3)$. Which of these points is closer to the line defined by $\{x = 2y, y = 2z\}$?

3. Least squares. **This problem is to be done by hand.** Consider the system of equations $Au = b$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

   (a) Show that there are no exact solutions for u.

   (b) Setup and find the solution $\hat{u}$ to the normal equations that minimizes the length of the vector $b - A\hat{u}$.

   (c) Find the QR factorization, $A = QR$ using the Gram-Schmidt procedure. Use it to solve for $\hat{u}$.

   (d) What best-fit problem does this least squares problem correspond to? Sketch the data points and the best-fit curve.

4. Linear transformations.

   (a) Consider a Householder matrix of form $H = I - 2uu^\top$ where u is a $n \times 1$ unit vector.

      i. Show that H is symmetric ($H^\top = H$), orthogonal ($H^\top H = I$), and that it is its own inverse ($H = H^{-1}$). Is H full rank?

      ii. Find all eigenvalues of H.

   (b) Consider the matrix $D = I - uu^\top$ where u is a $n \times 1$ unit vector.

      i. Describe the transformation of a vector v when multiplied on the left by the matrix D. Similarly, describe the transformation $D^{42}v$.

      ii. Is D full rank? If not, find an orthonormal basis for the null space of D.

      iii. Does D have a zero eigenvalue? If so, how many? Find the associated eigenvector.

5. Population dynamics. **This problem is to be done by hand.** A rabbit population (r) and wolf population (w) satisfy the following pair of coupled differential equations:

$$\frac{dr}{dt} = 6r - 2w, \tag{1}$$

$$\frac{dw}{dt} = 2r + w. \tag{2}$$

If the initial number of rabbits is 30 and the initial number of wolves is 30, what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?