# Problem Set 2 Solutions

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$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

Solve $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \begin{matrix} \rightarrow y_1 = 2 \\ \rightarrow y_1 + y_2 = 2 \rightarrow y_2 = 0 \\ \rightarrow -y_2 + y_3 = 2 \rightarrow y_3 = 2 \end{matrix}$$

Solve $Ux = y$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \begin{matrix} \rightarrow x_1 + x_3 = 2 \rightarrow x_1 = 1 \\ \rightarrow -x_2 + x_3 = 0 \rightarrow x_2 = 1 \\ \rightarrow 2x_3 = 2 \rightarrow x_3 = 1 \end{matrix}$$

$$\boxed{x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

2 Recall that $AA^{-1} = I$

So solving for $A^{-1}$ is equivalent to solving $n$-linear systems of the form $Ax_i = b_i$, where $b_i$ is the $i$th column of $I$.

In class, we saw that the LU-decomposition of $A = LU$ takes $\sim \frac{2}{3}n^3$ flops.

Also we saw that solving each $(LU)x_i = b_i$ takes $\sim 2n^2$ flops.

So the total number of flops is

$$\frac{2}{3}n^3 + (2n^2)\,n = \boxed{\frac{8}{3}n^3}$$

LU-decomp (only done once).

cost of solving one $LUx_i = b_i$

$n$-systems $(LUx_i = b_i)$ to solve

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(3) Taylor expand each term:

$$u_{i+2} = u_i + 2h u_i' + \frac{(2h)^2}{2} u_i'' + \frac{(2h)^3}{6} u_i''' + \frac{(2h)^4}{24} u_i'''' + \frac{(2h)^5}{120} u_i''''' + \ldots$$

$$u_{i+1} = u_i + h u_i' + \frac{h^2}{2} u_i'' + \frac{h^3}{6} u_i''' + \frac{h^4}{24} u_i'''' + \frac{h^5}{120} u_i''''' + \ldots$$

$$u_{i-1} = u_i - h u_i' + \frac{h^2}{2} u_i'' - \frac{h^3}{6} u_i''' + \frac{h^4}{24} u_i'''' - \frac{h^5}{120} u_i''''' + \ldots$$

$$u_{i-2} = u_i - 2h u_i' + \frac{(2h)^2}{2} u_i'' - \frac{(2h)^3}{6} u_i''' + \frac{(2h)^4}{24} u_i'''' - \frac{(2h)^5}{120} u_i''''' + \ldots$$

Combining:

$$\frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2h^3} =$$

$$\frac{u_i \cancelto{0}{(1-2+2-1)}}{2h^3} + \frac{h u_i' \cancelto{0}{(2-2-2+2)}}{2h^3} + \frac{h^2 u_i'' \cancelto{0}{\left(\frac{2^2}{2} - \frac{2}{2} + \frac{2}{2} - \frac{2^2}{2}\right)}}{2h^3}$$

$$+ \frac{h^3 u_i''' \left(\frac{2^3}{6} - \frac{2}{6} - \frac{2}{6} + \frac{2^3}{6}\right)}{2h^3} + \frac{h^4 u_i'''' \cancelto{0}{\left(\frac{2^4}{24} - \frac{2}{24} + \frac{2}{24} - \frac{2^4}{24}\right)}}{2h^3} + \frac{h^5 u_i''''' \left(\frac{2^5}{120} - \frac{2}{120} - \frac{2}{120} + \frac{2^5}{120}\right)}{2h^3} + \ldots$$

$$= u_i''' + \frac{h^2}{4} u_i''''' + \ldots$$

$$\Rightarrow \boxed{u_i''' = \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2h^3} + O(h^2)}$$

B1

4 $\quad \dfrac{-d^2u}{dx^2} + \dfrac{du}{dx} = 1 \qquad u(0) = 0$
$\qquad\qquad\qquad\qquad\qquad\qquad u(1) = 1$

a) $u(x) = x + A + Be^x$

$u(0) = 0 + A + B = 0 \rightarrow \boxed{A = -B}$

$u(x) = x - B + Be^x$

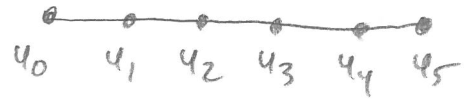
$u(1) = 1 - B + Be = 0 \rightarrow 1 = B(1-e)$

$\boxed{B = \dfrac{1}{1-e}}$

$$\boxed{u(x) = x + \frac{1}{1-e}\left(e^x - 1\right)}$$

B2)

$$-\frac{d^2u}{dx^2} + \frac{du}{dx} = 1 \qquad \begin{array}{l} u(0)=0 \\ u(1)=0 \end{array}$$




$h = 1/5$

b) $$\frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + \frac{u_{i+1} - u_{i-1}}{2h} = 1$$

Node 0: $u_0 = 0$

Interior Node $i$: $$u_{i-1}\underbrace{\left(\frac{-1}{h^2} - \frac{1}{2h}\right)}_{-55/2} + u_i \underbrace{\left(\frac{2}{h^2}\right)}_{50} + u_{i+1}\underbrace{\left(\frac{-1}{h^2} + \frac{1}{2h}\right)}_{-45/2} = 1$$

Node 5: $u_5 = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -55/2 & 50 & -45/2 & 0 & 0 & 0 \\ 0 & -55/2 & 50 & -45/2 & 0 & 0 \\ 0 & 0 & -55/2 & 50 & -45/2 & 0 \\ 0 & 0 & 0 & -55/2 & 50 & -45/2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

In MATLAB : $A\backslash b = u = \begin{bmatrix} 0 \\ 0.0714 \\ 0.1141 \\ 0.1220 \\ 0.0871 \\ 0 \end{bmatrix}$

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%True solution
x=linspace(0,1,1000);
u=x+1/(1-exp(1))*(exp(x)-1);

%Centered difference solution
A=zeros(6,6);
b=zeros(6,1);
A(1,1)=1;
A(6,6)=1;
for i=2:5
    A(i,i-1)=-55/2;
    A(i,i)=50;
    A(i,i+1)=-45/2;
    b(i)=1;
end

xi=0:0.2:1;
ui=A\b;

plot(x,u,'-b',xi,ui,'or');
xlabel('x');
ylabel('u');
legend('True solution','Centered difference');
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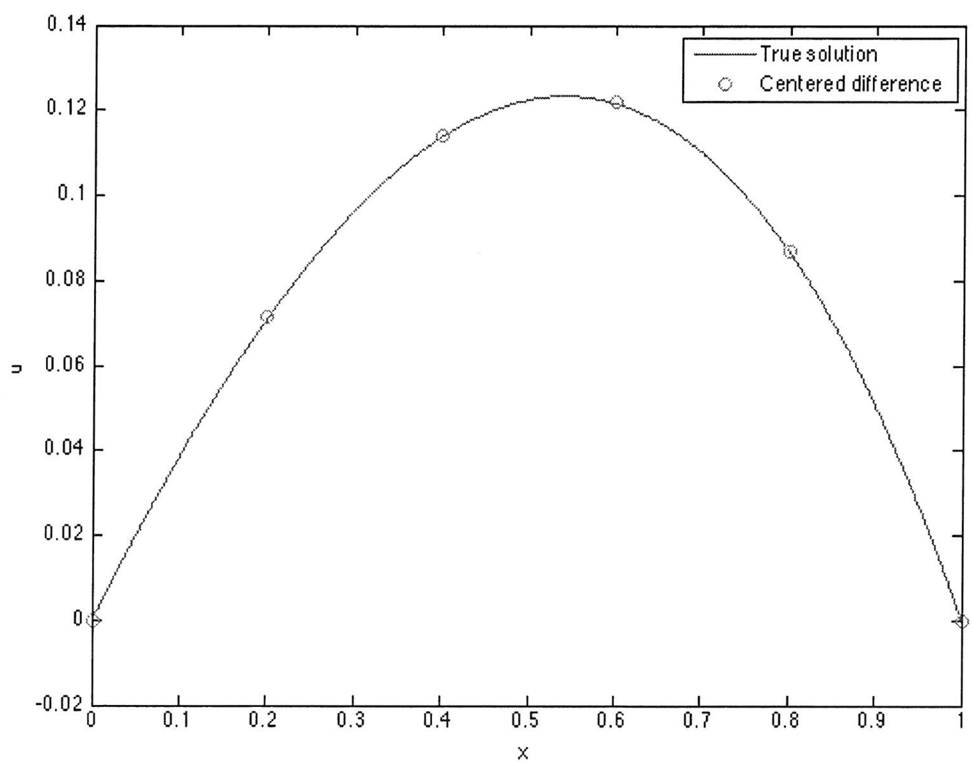

(5) $\frac{d^2u}{dx^2} = f(x)$ $\quad u'(0)=0$, $u'(1)=0$

a) $\int_0^1 \frac{d^2u}{dx^2}\,dx = \int_0^1 f(x)\,dx$

$\cancel{\left.\frac{du}{dx}\right|_{x=1}}^{0} - \cancel{\left.\frac{du}{dx}\right|_{x=0}}^{0} = \int_0^1 f(x)\,dx \Rightarrow \boxed{\int_0^1 f(x)\,dx = 0}$

Physically this means there is no net force on the string.

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b) $\int_0^1 \delta\left(x-\frac{1}{3}\right) - \delta\left(x-\frac{2}{3}\right)dx =$

$\int_0^1 \delta\left(x-\frac{1}{3}\right)dx - \int_0^1 \delta\left(x-\frac{2}{3}\right)dx = \underbrace{\int_{-\infty}^{\infty} \delta\left(x-\frac{1}{3}\right)dx}_{1} - \underbrace{\int_{-\infty}^{\infty} \delta\left(x-\frac{2}{3}\right)dx}_{1}$

$= 1 - 1 = 0 \rightarrow \boxed{\int_0^1 f(x)\,dx = 0}$

(2)

c) $f(x) = \delta\left(x-\frac{1}{3}\right) - \delta\left(x-\frac{2}{3}\right)$

Linear function for $\left.\begin{array}{l} 0 \le x < \frac{1}{3} \\ \frac{1}{3} < x < \frac{2}{3} \\ \frac{2}{3} < x \le 1 \end{array}\right\}$ since $f(x) = 0$ for $x \neq \frac{1}{3}, \frac{2}{3}$

Continuity $\rightarrow$ $u\left(\frac{1}{3}\right)^- = u\left(\frac{1}{3}\right)^+$, $u\left(\frac{2}{3}\right)^- = u\left(\frac{2}{3}\right)^+$

Jump conditions $\rightarrow$ $\left[\frac{du}{dx}\right]\left(\frac{1}{3}\right) = 1$, $\left[\frac{du}{dx}\right]\left(\frac{2}{3}\right) = -1$

$$u(x) = \begin{cases} ax + b & \text{for } 0 \le x < \frac{1}{3} \\ cx + d & \text{for } \frac{1}{3} < x < \frac{2}{3} \\ ex + g & \text{for } \frac{2}{3} < x \le 1 \end{cases}$$

Boundary conditions $\rightarrow$ $u'(0) = 0 \Rightarrow a = 0$

$u'(1) = 0 \Rightarrow e = 0$

Continuity $\rightarrow$ $b = \frac{1}{3}c + d$, $\frac{2}{3}c + d = g$

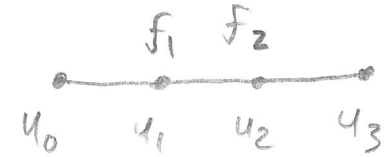
Jump conditions $\rightarrow$ $\left.\begin{array}{l} c - a = 1 \Rightarrow c = 1 \\ e - c = -1 \Rightarrow c = 1 \end{array}\right\}$ $\rightarrow$ $\begin{cases} b = \frac{1}{3} + d \\ g = \frac{2}{3} + d \end{cases}$

$$\boxed{u(x) = \begin{cases} \frac{1}{3} + d & \text{for } 0 \le x < \frac{1}{3} \\ x + d & \text{for } \frac{1}{3} < x < \frac{2}{3} \\ \frac{2}{3} + d & \text{for } \frac{2}{3} < x \le 1 \end{cases}}$$

infinitely many solutions (valid for any $d$)

(2)

d) $f(x) = \delta\left(x - \frac{1}{3}\right) - \delta\left(x - \frac{2}{3}\right)$




$h = 1/3$

Node 0: $\dfrac{u_1 - u_0}{h} = 0$

Node 1: $\dfrac{u_2 - 2u_1 + u_0}{h^2} = 3$ since we want $f_1 h = 1$

Node 2: $\dfrac{u_3 - 2u_2 + u_1}{h^2} = -3$ since we want $f_2 h = -1$

Node 3: $\dfrac{u_3 - u_2}{h} = 0$

$$\begin{bmatrix} -1/h & 1/h & 0 & 0 \\ 1/h^2 & -2/h^2 & 1/h^2 & 0 \\ 0 & 1/h^2 & -2/h^2 & 1/h^2 \\ 0 & 0 & -1/h & 1/h \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 & 0 & 0 \\ 9 & -18 & 9 & 0 \\ 0 & 9 & -18 & 9 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -3 \\ 0 \end{bmatrix}$$

rank = 3

Notice each row sums to zero, so $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in N(A)$.

We need a particular solution as well to get all solutions →

One example is $u_p = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

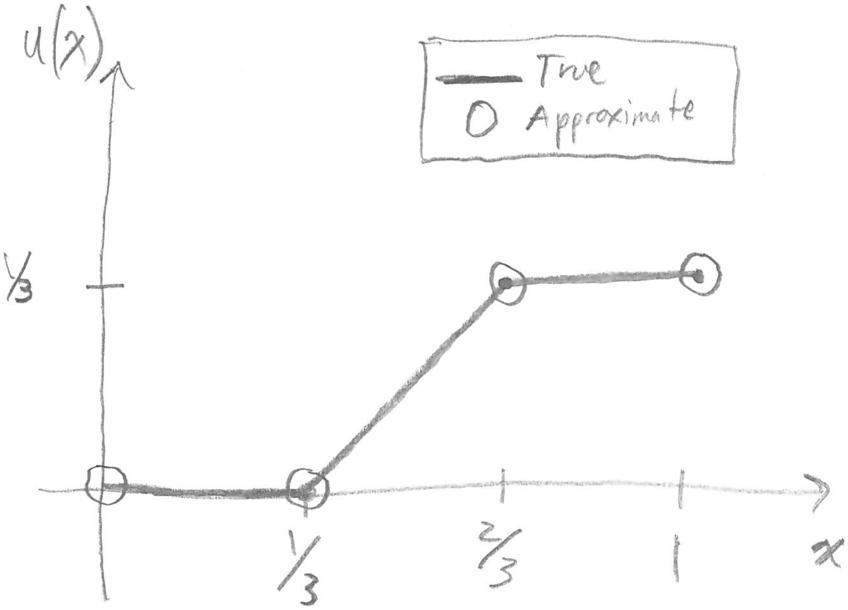
So the complete solution is ⟶ $\boxed{u = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$

c3

e) $u(0)=0$ is satisfied when $d=\frac{-1}{3}$ in both cases:

$$\text{True}: \quad u(x) = \begin{cases} 0 & \text{for } 0 \le x < \frac{1}{3} \\ x - \frac{1}{3} & \text{for } \frac{1}{3} < x < \frac{2}{3} \\ \frac{1}{3} & \text{for } \frac{2}{3} < x \le 1 \end{cases}$$

$$\text{Approximate}: \quad u = \begin{bmatrix} 0 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix}$$




The centered difference in this case is exact!

This is because → Our delta functions in $f(x)$ align with nodes.

→ The solution between nodes is linear so our difference approximation is exact, and our second difference corresponds to the jump condition.