Problem Set 2

Unless otherwise specified, you may use MATLAB to assist with computations. Please provide a print-out of the code used and its output with your assignment.

1. **This problem is to be done by hand.** Find the unique solution to the following linear system by LU decomposition and back-substitution.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \tag{1}$$

2. The inverse A^{-1} of a non-singular $n \times n$ matrix A can be calculated using the LU decomposition of A. Describe an algorithm for computing A^{-1} by solving n systems of equations and show that its asymptotic (large n) operation count is $\sim \frac{8}{3}n^3$.

3. Show that the third derivative can be approximated by the following difference:

$$\frac{d^3 u_i}{dx^3} \approx \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2h^3} \tag{2}$$

   where h is the spacing between neighboring grid points. What is the leading error in this approximation?

4. Consider the following boundary value problem:

$$-\frac{d^2 u}{dx^2} + \frac{du}{dx} = 1 \tag{3}$$

   with boundary conditions $u(0) = 0$ and $u(1) = 0$.

   (a) The true solution $u(x)$ is the particular solution $u = x$ plus any $A + Be^x$. Which A and B satisfy the boundary conditions?

   (b) Setup and solve the centered difference approximation to the boundary value problem. Use a grid spacing of $h = 1/5$. Compare the approximate solution to the true solution by plotting them on the same axes.

5. Consider the following boundary value problem:

$$\frac{d^2 u}{dx^2} = f(x) \tag{4}$$

   with boundary conditions $u'(0) = 0$ and $u'(1) = 0$.

   (a) Under what condition on $f(x)$ does this problem have a solution?

   (b) Show that $f(x) = \delta\left(x - \frac{1}{3}\right) - \delta\left(x - \frac{2}{3}\right)$ satisfies this condition, and thus that the problem has a solution.

(c) Find *all* solutions with $f(x) = \delta\left(x - \frac{1}{3}\right) - \delta\left(x - \frac{2}{3}\right)$.

(d) Setup the centered difference approximation to the boundary value problem with $f(x) = \delta\left(x - \frac{1}{3}\right) - \delta\left(x - \frac{2}{3}\right)$ and grid spacing $h = 1/3$. (Hint: when discretizing each delta function, choose the magnitude of the forcing at the node carefully so that it still corresponds to a unit load in the integral sense). Find *all* solutions.

(e) Plot the true (found in part (c)) and approximate (found in part (d)) solutions which satisfy the additional condition $u(0) = 0$. How accurate is the approximate solution? Why does it do so well in this problem?