18.085 : Problem Set 1 Solutions

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a) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow[\text{+ row(1)}]{\text{row(2)}} \overset{\text{ref}}{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}$

1 pivot $\rightarrow$ rank $=1$ $\Rightarrow$

$$\boxed{\begin{aligned} &\dim(C(A)) = \dim(C(A^T)) = 1 \\ &\dim(N(A)) = n - r = 1 \\ &\dim(N(A^T)) = m - r = 1 \end{aligned}}$$

Neither

b) $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

Already in ref form.

3 pivots $\rightarrow$ rank $= 3$ $\Rightarrow$

$$\boxed{\begin{aligned} &\dim(C(A)) = \dim(C(A^T)) = 3 \\ &\dim(N(A)) = n - r = 3 \\ &\dim(N(A^T)) = m - r = 0 \end{aligned}}$$

Full row rank

c) $\underset{n\times 1}{v}\ \underset{1\times n}{v^T} \rightsquigarrow n\times n$ matrix

Columns are linear multiples of v only $\rightarrow$ rank $=1$ $\Rightarrow$

$$\boxed{\begin{aligned} &\dim(C(vv^T)) = \dim(C((vv^T)^T)) = 1 \\ &\dim(N(vv^T)) = \dim(N((vv^T)^T)) = n-1 \end{aligned}}$$

Neither

d) $\underset{1\times n}{u^T}\ \underset{n\times 1}{u} \rightarrow 1\times 1$ matrix $\rightarrow$ rank $=1$ $\Rightarrow$

$$\boxed{\begin{aligned} &\dim(C(u^Tu)) = \dim(C((u^Tu)^T)) = 1 \\ &\dim(N(u^Tu)) = \dim(N((u^Tu)^T)) = 0 \end{aligned}}$$

Both

Note: $u^Tu = \overline{u}\cdot\overline{u}$

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e) $\underset{n\times m}{A^T}\underset{m\times n}{A} \rightarrow n\times n$ matrix

A is full-column rank, so $\text{rank}(A)=n$ $\rightarrow$ A has independent columns
$\rightarrow$ A has n-independent rows

Since the rows of A^TA will be a linear combo of the rows of A, the rank will also be n. $\Rightarrow$

$$\boxed{\begin{array}{l} \dim(C(A^TA)) = \dim(C((A^TA)^T)) = n \\ \dim(N(A^TA)) = \dim(N((A^TA)^T)) = 0 \end{array}}$$

Both

f) $\underset{m\times n}{B}\underset{n\times m}{B^T} \rightarrow m\times m$ matrix

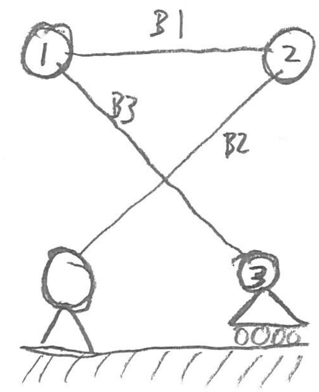
B is full-row rank, so $\text{rank}(B)=m$ $\rightarrow$ B has independent rows
$\rightarrow$ B has m-independent columns.

Since the cols of BB^T will be a linear combo of the cols of B, the rank will also be m. $\Rightarrow$

$$\boxed{\begin{array}{l} \dim(C(BB^T)) = \dim(C((BB^T)^T)) = m \\ \dim(N(BB^T)) = \dim(N((BB^T)^T)) = 0 \end{array}}$$

Both

2 a)



Degrees of freedom are: $u_1^H, u_1^V, u_2^H, u_2^V, u_3^H$

Bars are: B1, B2, B3

Bar 1: $\theta = 0 \rightarrow \cos\theta = 1$
$\sin\theta = 0$

Bar 2: $\theta = \pi/4 \Rightarrow \cos\theta = 1/\sqrt{2}$
$\sin\theta = 1/\sqrt{2}$

Bar 3: $\theta = \pi/4 \rightarrow \cos\theta = 1/\sqrt{2}$
$\sin\theta = 1/\sqrt{2}$

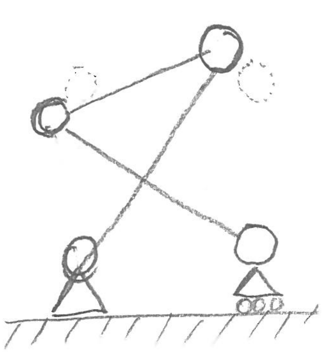
$$\boxed{\begin{array}{c} B1: \\ B2: \\ B3: \end{array} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

b) $\dim(N(A)) = \#$ of independent deformations $= n - r = 5 - 3 = \boxed{2}$

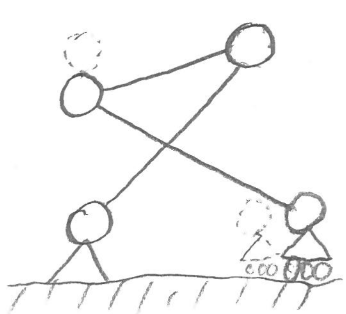
(In MATLAB: rank(A) = 3)

c) In MATLAB: null(A, 'r') =

$$\begin{bmatrix} -1 & 0 \\ -1 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{array}{cc} \text{Mode 1} & \text{Mode 2} \end{array}$$



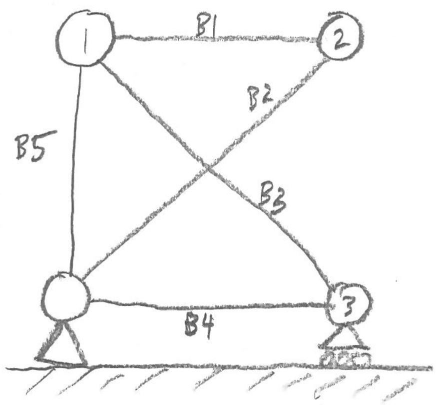
Mode 1



Mode 2

d) For stability we need rank$(A) = \#$ DOFs, so we need at minimum 2 more bars.

e) A stable truss:



$$\begin{array}{c} B1: \\ B2: \\ B3: \\ B4: \\ B5: \end{array} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Bar 4: $\theta = 0 \Rightarrow \cos\theta = 1$
$\sin\theta = 0$

Bar 5: $\theta = \pi/2 \rightarrow \cos\theta = 0$
$\sin\theta = 1$

In MATLAB: rank$(A) = 5$

$\Rightarrow \dim(N(A)) = 5 - 5 = 0 \Rightarrow$ no deformation modes that preserve bar length.

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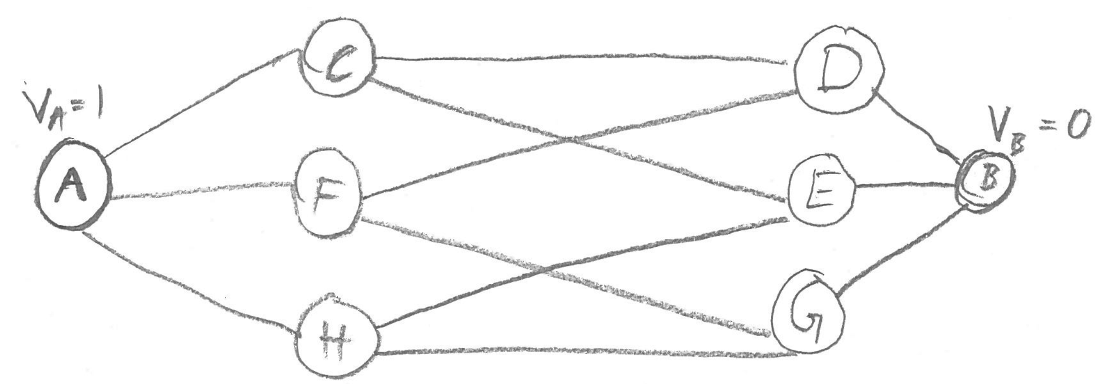
(3)



Node (A): $V_A = 1$

Node (B): $V_B = 0$

Node (C): $(V_A - V_C) + (V_D - V_C) + (V_E - V_C) = 0$

Node (D): $(V_B - V_D) + (V_C - V_D) + (V_F - V_D) = 0$

Node (E): $(V_B - V_E) + (V_C - V_E) + (V_H - V_E) = 0$

Node (F): $(V_A - V_F) + (V_D - V_F) + (V_G - V_F) = 0$

Node (G): $(V_B - V_G) + (V_F - V_G) + (V_H - V_G) = 0$

Node (H): $(V_A - V_H) + (V_E - V_H) + (V_G - V_H) = 0$

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}
\begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -3 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & -3 \end{bmatrix} \end{array}
\begin{bmatrix} V_A \\ V_B \\ V_C \\ V_D \\ V_E \\ V_F \\ V_G \\ V_H \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In MATLAB: $A \backslash b = \begin{bmatrix} 1 \\ 0 \\ 3/5 \\ 2/5 \\ 2/5 \\ 3/5 \\ 2/5 \\ 3/5 \end{bmatrix}$ $\longrightarrow$ current into node B $= \dfrac{(V_D - \cancel{V_B})}{R} + \dfrac{(V_E - \cancel{V_B})}{R} + \dfrac{(V_G - \cancel{V_B})}{R}$

$$= \frac{2}{5R} + \frac{2}{5R} + \frac{2}{5R} = \frac{6}{5R}$$

$$R_{eq} = \frac{\Delta V}{I_B} = \frac{1}{(6/5R)} = \boxed{\frac{5}{6} R}$$

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(4) a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\longrightarrow$ By inspection $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution, but is it the only solution?

check rank $\rightarrow$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[-\text{row}(2)]{\text{row}(3)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow$ 2 pivots, $r=2$, $\dim(N(A)) = 1$

Thus there are infinitely many solutions of form $x = x_p + x_n$

$x_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, Notice: column (3) = col (1) + col (2) $\rightarrow$ $x_n = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$0 =$ col (1) + col (2) − col (3)

$$\boxed{x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}$$

b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ we already saw that rank$(A) < n$, so we either have no or infinitely-many solutions.

Q: Is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in C(A)$? $\rightarrow$ Note that in all columns the 2nd element = 3rd element

$\rightarrow$ No combination of these vectors can thus produce $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

$\rightarrow$ **No solution!**

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c) $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

check rank $\rightarrow$ $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[-\text{row}(2)]{\text{row}(3)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ rank $= 3 = n$, Matrix is invertible! So we have a unique soln.

Gaussian Elimination: $$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[-\text{row}(2)]{\text{row}(3)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[-\text{row}(3)]{\text{row}(2)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[-\text{row}(3)]{\text{row}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{aligned} x_1 &= 2 \\ x_2 &= 2 \\ x_3 &= -1 \end{aligned}$$

$$\boxed{x = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}$$