YOUR NAME: ____________________________________________

YOUR SCORE: _________ / 100 + ___________ BONUS

This exam has 5 questions!
Question 1 (5+8+3+3+6=25pts) Real and complex Fourier series.

Let \( f(x) = |x| \) be defined on \( -\pi \leq x < \pi \), and let \( g(x) \) be its \( 2\pi \)-periodic extension.

(a) Sketch the function \( g(x) \). Label its maximum value(s), and the location(s) of the maximum value(s), on your graph.

(b) Compute the real Fourier coefficients of \( g(x) \). You may use the symmetry of the function to reduce your workload, and use integration by parts.
(c) What are the complex Fourier coefficients of $g(x)$? You may use your answer from part (b) to reduce your workload.

(d) How fast do the Fourier coefficients of $g(x)$ decay as $n \to \infty$? Explain why you could have predicted this without doing parts (b) or (c).

(e) Use Parseval’s theorem to deduce the following equality:

$$\sum_{n \geq 1, \text{odd}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$$

Let \( w = e^{i2\pi/N} \).

(a) The entries of the Fourier matrix \( F \) are \( F_{jk} = \) ____________.

(b) The entries of the inverse of the Fourier matrix \( F^{-1} \) are \( F^{-1}_{jk} = \) ____________.

Let \( y(x) \) be the following function (assumed to be \( 2\pi \)-periodic):
\[
y(x) = \begin{cases} 
x & 0 \leq x \leq \pi \\
2\pi - x & \pi \leq x \leq 2\pi.
\end{cases}
\]

(c) Sample \( y(x) \) at the \( N = 3 \) Fourier sample points to get the sample vector \( \vec{y} = (y(0), \cdots) \):
\[
\vec{y} = (\text{__________________________})^T.
\]

(d) Find the discrete Fourier coefficients \( \vec{c} \) of \( \vec{y} \) by solving the equation \( \vec{y} = F\vec{c} \), again for \( N = 3 \). The answer uses numbers, no \( w \)'s and no square roots! You might want to recall that columns and rows of \( F \) sum to 0, except for the 0th row and column.
Question 3 (6+6+4=16 pts) The Discrete Fourier Transform Part 2.

You are given the vector \( \vec{c} = \frac{1}{3}(2, -1, -1)^T \) (different from part 1).

(a) Take the cyclic convolution of the vector \( \vec{c} \) with itself: \( \vec{c} \ast \vec{c} \). You can use the direct method you like ("long multiplication" or using a circulant matrix), but DO NOT use the convolution rule.

(b) Multiply the Fourier matrix \( F \) (for \( N = 3 \)) with the convolution of (a). That is, calculate \( F(\vec{c} \ast \vec{c}) \). The answer uses numbers, not \( w \)'s.

(c) Let \( \vec{y} = F\vec{c} = (0, 1, 1)^T \), same \( \vec{c} \) as above. Is the following equation true? YES or NO. No need to justify, just circle your answer.

\[
\begin{pmatrix}
  y_1 y_1 \\
  y_2 y_2 \\
  y_3 y_3 \\
\end{pmatrix}
= F((F^{-1} \vec{y}) \oplus (F^{-1} \vec{y})).
\]
Question 4 \((5 + 3 + 6 + 6 + 3 = 23 \text{ pts} + 5 \text{ BONUS})\) Fourier Integral Transform.

This question uses the Fourier Integral Transform. Let \(f(x)\) be the following function, defined over the whole real line \(\mathbb{R}\) (NOT \(2\pi\)-periodic). It looks like a hat.

\[
f(x) = \begin{cases} 
  x & 0 \leq x \leq \pi \\
  2\pi - x & \pi \leq x \leq 2\pi \\
  0 & \text{elsewhere.} 
\end{cases}
\]

(a) Draw functions \(f(x)\), \(g(x) = df/dx\) and \(h(x) = d^2f/dx^2\) on 3 separate graphs. Make sure to do this right, it’ll be useful in (b), (d) and maybe even (f).

(b) The solution \(u(x)\) to equation (1) (no calculation needed) is \(u(x) = \quad \). \(\quad \)

\[
\frac{-d^2u}{dx^2} = \delta(x) - 2\delta(x - \pi) + \delta(x - 2\pi)
\]

(c) Take the Fourier Integral Transform (FIT) of equation (1) to find an expression for \(\hat{u}(k)\), the FIT of \(u(x)\). The answer \(\hat{u}(k)\) involves only numbers and \(k\)’s, which means you’ll need to calculate the FIT of those delta functions.
Question 4 continued. \((5 + 3 + 6 + 6 + 3 = 23 \text{ pts} + 5 \text{ BONUS})\)

(d) Find the FIT \(\hat{g}(k)\) of \(g(x) = \frac{df}{dx}\) using the formula \(\hat{g}(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx}dx\).

(e) From your answer in (d), find the FIT \(\hat{f}(k)\) of \(f(x)\).

You can check your answer using (b) and (c) if you want! Only partial credit awarded if you use (b) and (c) but not (d) to find \(\hat{f}(k)\).

(f) BONUS QUESTION. OPTIONAL. Find the FIT \(\hat{h}(k)\) of \(h(x)\). You may use either the work you did in (c) or in (d) (or use both to check your answer if you’d like).
Question 5 (6 + 6 + 5 + 3 = 20 pts) Filtering.

Consider the following filter, which takes an infinite vector input $\vec{x}$ and outputs the infinite vector $\vec{y}$ as such:

$$y_k = x_k - \frac{1}{2}x_{k-1} - \frac{1}{2}x_{k-2}, \quad -\infty \leq k \leq \infty, \quad k \text{ integer.}$$

(a) Let $\vec{h}$ be the infinite vector such that the convolution $\vec{y} = \vec{h} \ast \vec{x}$ holds. It has only 3 non-zero components. Write down these 3 components (value + which $k$).

(b) Let the input have a frequency of $\omega$: $x_k = e^{jk\omega}$. Calculate the output $y_k$. Also, write the output as $y_k = x_k H(\omega)$. What is this function $H(\omega)$?

(c) Fill the blanks: $H(0) =$ ___________ and $H(\pi) =$ ___________. (If you couldn’t do (b), you might still be able to try this one out.)

(d) Is this filter LOWPASS or HIGHPASS (circle one)? No justification needed.