18.085: EXAM 2

July 28, 2014

NAME:

SCORE:
Question 1. (25 pts.)

Consider the following plane square truss:

(a) (5 pts.) Write down the matrix $A$ for this truss.

(b) (3 pts.) What is the rank of $A$?

*Hint:* It is probably easiest to determine the number of independent rows.
(c) (5 pts.) How many instabilities (rigid motions and mechanisms) does this truss have? Write down the corresponding displacement vector(s) $\vec{u}$.

*Hint:* Recall from earlier in the course that

\[
\text{rank}(A) + (\text{number of independent solutions to } A\vec{u} = \vec{0}) = \text{number of columns in } A
\]

(d) (3 pts.) Now fix nodes 1 and 2. Write down the matrix $A_1$ corresponding to this new truss.
(e) (4 pts.) What is the rank of $A_1$, and how many instabilities does the truss have? Write down the corresponding displacement vector(s) $\vec{u}$.

(f) (5 pts.) For the truss in part (d) (with nodes 1 and 2 fixed), add a bar between nodes 1 and 4. Write down the new matrix $A_2$. How many mechanisms does this truss have? Explain your answer mathematically; do not give purely physical arguments!
Question 2. (10 pts.)

Consider the equation

\[ x^3 + 4 = 2x \]

Use Newton’s method to find an approximate solution to this equation. Specifically, carry out two iterations of Newton’s method starting with the initial guess \( x_0 = 0 \).
Question 3. (25 pts.)

We consider a non-uniform beam of length 2 hanging under the force of gravity. Specifically, we will solve the equation

\[
\frac{d^2}{dx^2} \left( c(x) \frac{d^2 u}{dx^2} \right) = f(x), \quad 0 \leq x \leq 2
\]

The force is \( f(x) = -1 \), and the stiffness function \( c(x) \) is

\[
c(x) = \begin{cases} 
1 & \text{if } x \leq 1 \\
x & \text{if } x > 1 
\end{cases}
\]

We assume the beam to be simply supported, so we take the boundary conditions

\[
u = 0, \quad M = 0
\]

at both ends, where \( M(x) = c(x) \frac{d^2 u}{dx^2} \) is the bending moment.

(a) (10 pts.) Find the bending moment \( M(x) = c(x) \frac{d^2 u}{dx^2} \). (Don’t forget to impose the boundary conditions on \( M \).)
(b) (5 pts.) Find \( u'' \).

(c) (10 pts.) Find the displacement \( u(x) \) by imposing the boundary conditions on \( u(x) \). Also impose the requirement that \( u \) and \( u' \) must be continuous along the beam. Once you impose these conditions, you should obtain equations that determine the constants in \( u(x) \). You do not have to solve the equations.
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Question 4. (15 pts.)

Solve Laplace’s equation $\Delta u = 0$ on the unit disc with the boundary condition

$$\frac{\partial u}{\partial r} = \cos 2\theta - \sin 4\theta$$

on the boundary of the disc.
Question 5. (25 pts.) In this question, we will set up the finite element method for the equation

\[- \frac{d}{dx} \left( c(x) \frac{du}{dx} \right) = \delta(x - 2/3), \quad 0 \leq x \leq 1\]

where

\[c(x) = \begin{cases} 
2 & \text{if } x < 1/3 \\
4 & \text{if } x > 1/3 
\end{cases}\]

We take the boundary conditions

\[w(0) = u(1) = 0\]

where \(w(x) = c(x) \frac{du}{dx}\).

(a) (5 pts.) Write down the weak form of the differential equation. What condition must the test functions \(v(x)\) satisfy?

(b) (5 pts.) Take \(h = 1/3\). Draw the hat functions you will use to solve this problem.
(c) (10 pts.) Construct the matrix $K$ for this problem. Show your work.
(d) (5 pts.) Construct the vector $\vec{F}$ for this problem. Show your work.

(e) **Bonus (5 pts.):** Solve the equation $K\vec{U} = \vec{F}$ for $\vec{U}$ and write down the finite element solution $U(x)$. Graph the solution.