18.085: PROBLEM SET 7 SOLUTIONS

Due August 4, 2014 9:30AM IN CLASS

Question 1. (35 pts.) Solving Poisson's equation numerically.

We will solve Poisson's equation

$$-\Delta u = 6(x-y)+2, \quad 0 \le x \le 1, 0 \le y \le 1$$

on the unit square, subject to the boundary conditions

- $u = x^2(1-x)$ on the horizontal edges of the square, and
- $u = y(1-y)^2$ on the vertical edges.

(a) Verify that $u(x,y) = x^2(1-x) + y(1-y)^2$ is the exact solution to this problem. To visualize this solution in Matlab, use the following few lines of code and print the plot:

```
[X,Y] = meshgrid(0:.001:1);
U_exact = (X.^2).*(1-X) + Y.*((1-Y).^2);
surf(X,Y,U_exact,'edgecolor','none');
view([0,90]); colorbar; axis square;
```

(b) We will now solve the equation numerically in Matlab. Let N be the number of unknowns in each direction (x and y). Number the nodes by row from bottom to top, as we did in class. Write a code that follows these steps:

i. Construct the matrix K, the 1D second-derivative matrix, using the following few lines of code:

```
v = [2 -1 zeros(1,N-2)];
K = toeplitz(v);
```

ii. Construct the matrix $K2D$, the 5-point Laplacian matrix in 2D, using K.
Hint: Equation (3) on page 284 of the textbook is useful.

iii. Now construct the vector $\vec{F}$ using the right-hand side of Poisson's equation, ignoring the boundary conditions for now. (Careful with the indexing!)

iv. As we saw in class, the boundary conditions modify some of the entries of $\vec{F}$. For now, ignore the four corners; we will deal with them in the next step. For example, we can modify $\vec{F}$ using the boundary conditions on the bottom edge as follows:

```
for k = 2:N-1
    F(k) = F(k) + g(k*h,0);
end;
```

In this code, $g(x, y)$ is the boundary value of $u(x, y)$, and h is the grid spacing. Do the same for the other edges. (Careful on how h relates to N.)

v. Now deal with the four corners. For example, the value of $\vec{F}$ corresponding to the bottom left corner is modified as follows:

```
F(1) = F(1) + g(h,0) + g(0,h);
```

Do the same for the other three corners.

vi. Solve for the unknowns $\vec{U}$ using backslash.

vii. To visualize your solution, you will need to rearrange the components of your vector $\vec{U}$ in matrix form. To do so, use the following few lines of code:

```
UMat = reshape(U,N,N)';
[X2,Y2] = meshgrid(h:h:1-h);
surf(X2,Y2,UMat,'edgecolor','none');
view([0,90]); axis square; colorbar;
```

(c) Show plots of your approximate solutions for $N = 10$, 30 and 50. How do they compare (visually) to the exact solution you found in part (a)?

SOLUTION:

(a) For $u(x, y) = x^2(1 - x) + y(1 - y)^2$, it is clear that $u(x, 0) = u(x, 1) = x^2(1 - x)$ and $u(0, y) = u(1, y) = y(1 - y)^2$, so u satisfies the boundary conditions. We also find that

$$u_x = 2x - 3x^2, \quad u_{xx} = 2 - 6x$$

$$u_y = 3y^2 - 4y + 1, \quad u_{yy} = 6y - 4$$

so we obtain

$$-\Delta u = -(u_{xx} + u_{yy}) = 6x - 6y + 2$$

as desired. A plot of the exact solution is shown in Figure 1.

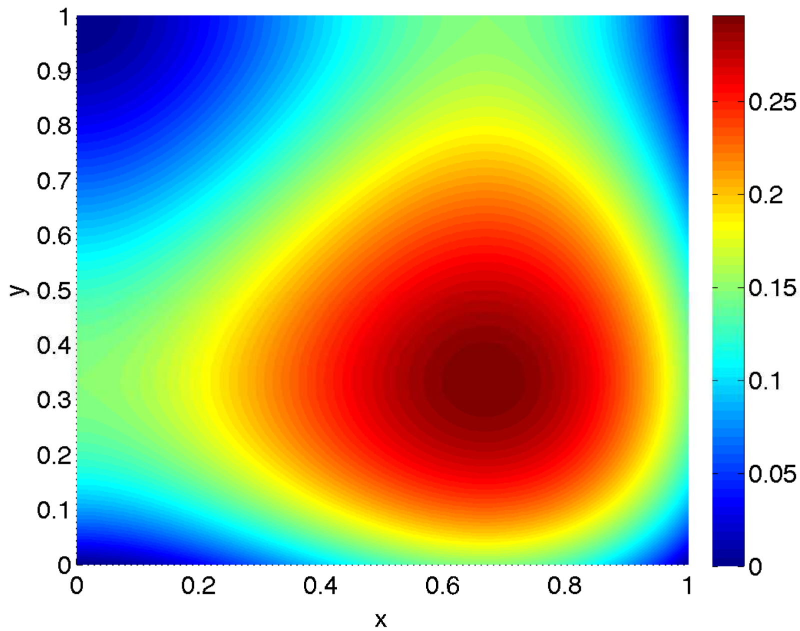

FIGURE 1. Plot of the exact solution to the problem in Question 1.

(b) The code used to solve the equation is given below:

```
N = 10;
dx = 1/(N+1);
v = [2 -1 zeros(1,N-2)];
K = toeplitz(v);
K2D = kron(eye(N),K) + kron(K,eye(N));
Fint = zeros(1,N^2);
Fbound = zeros(1,N^2);

% deal with interior
j = 1;
for l = 1:N
    for k = 1:N
        Fint(j) = dx^2*func(k*dx,l*dx);
        j = j+1;
```

```
    end;
end;

% deal with edges
for k = 2:N-1
    Fbound(k) = g(k*dx,0);
    Fbound((N-1)*N+k) = g(k*dx,1);
    Fbound(k*N-N+1) = g(0,k*dx);
    Fbound(k*N) = g(1,k*dx);
end;

% deal with corners
Fbound(1) = g(dx,0) + g(0,dx);
Fbound(N) = g(1-dx,0) + g(1,dx);
Fbound((N-1)*N+1) = g(0,1-dx) + g(dx,1);
Fbound(N^2) = g(1-dx,1) + g(1,1-dx);

Fnew = Fint + Fbound;

U = K2D\(Fnew');

[X2,Y2] = meshgrid(dx:dx:1-dx);
UMat = reshape(U,N,N)';
figure(1); surf(X2,Y2,UMat,'edgecolor','none'); view([0,90]);
title(['N = ' num2str(N)]); xlim([0 1]); ylim([0 1]);
xlabel('x','FontSize',16)
ylabel('y','FontSize',16)
colorbar
set(gca,'FontSize',16)

function out = func(x,y) % right side of Poisson equation
out = 6*(x-y) + 2;

function out = g(x,y) % boundary value of u
if (x == 0 || x == 1)
    out = y*(1-y)^2;
else
    out = x^2*(1-x);
end;
```

(c) The approximate solutions for $N = 10$, 30 and 50 are shown in Figure 2. Note that they approximate the exact solution from part (a) better and better as N gets larger.

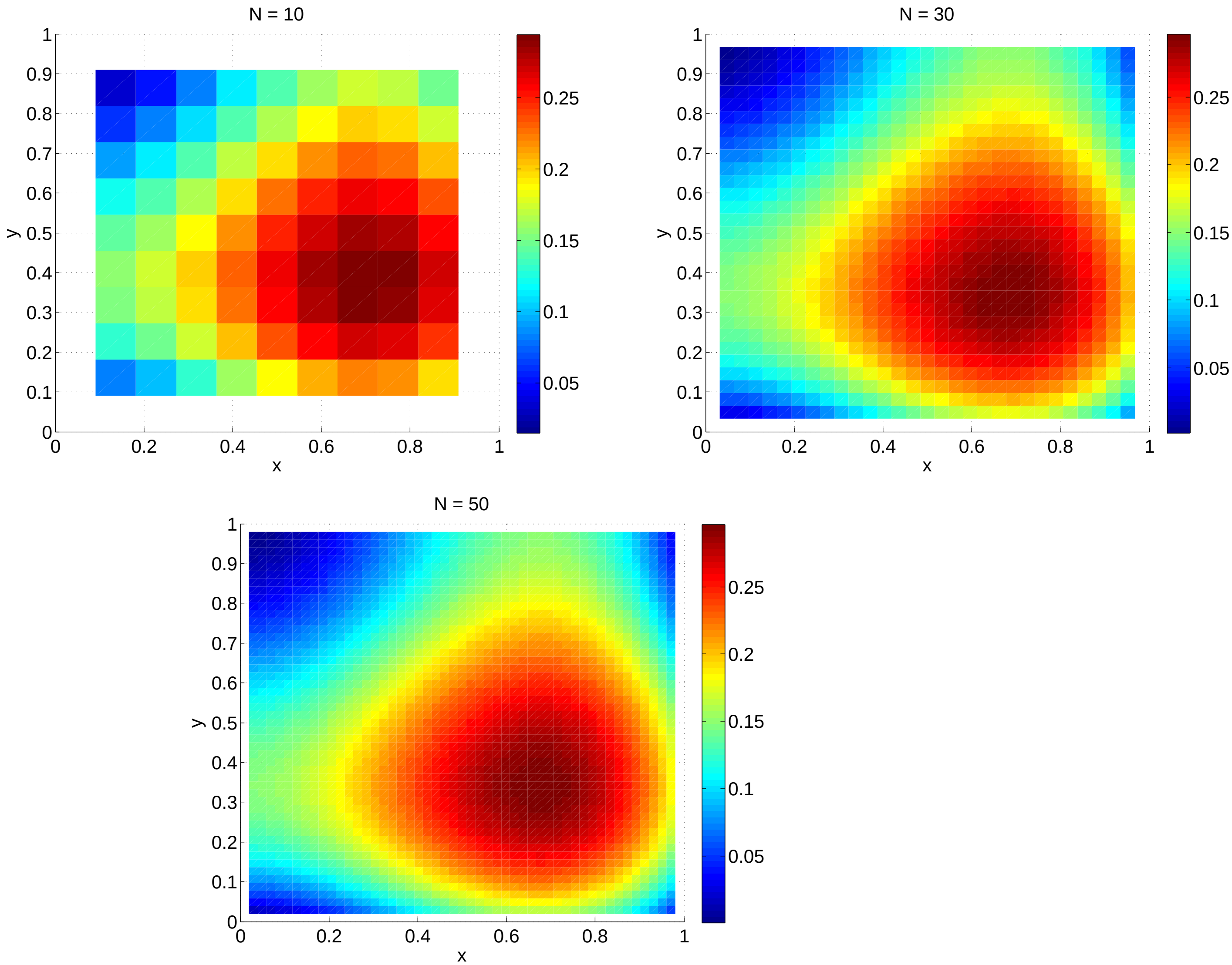

FIGURE 2. Plot of the approximate solutions for $N = 10$, 30 and 50, as needed in Question 1(c).

Question 2. (15 pts.) 9-point stencil for the Laplacian.

In class, we showed that the 5-point approximation for the negative Laplacian $-\Delta$ is given by

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \approx \frac{1}{h^2}\left(4u(x,y) - u(x+h,y) - u(x-h,y) - u(x,y+h) - u(x,y-h)\right)$$

where h is the grid spacing. This approximation is accurate to order $O(h^2)$. It turns out that the following 9-point approximation to the Laplacian may be more accurate:

$$\begin{aligned} -\Delta u \quad &\approx \quad \frac{1}{6h^2}\left[20u(x,y) - 4\left(u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)\right)\right. \\ &\qquad \left. - \left(u(x+h,y+h) + u(x+h,y-h) + u(x-h,y+h) + u(x-h,y-h)\right)\right] \end{aligned}$$

Suppose you want to solve Poisson's equation $-\Delta u = f$ on the unit square ($0 \le x \le 1$, $0 \le y \le 1$) with some boundary conditions. Write down the matrix K corresponding to the 9-point Laplacian for $h = 1/4$.

SOLUTION: The matrix K (with the factor of $1/6h^2$) looks like

$$K = \frac{8}{3}\begin{pmatrix} 20 & -4 & 0 & -4 & -1 & 0 & 0 & 0 & 0 \\ -4 & 20 & -4 & -1 & -4 & -1 & 0 & 0 & 0 \\ 0 & -4 & 20 & 0 & -1 & -4 & 0 & 0 & 0 \\ -4 & -1 & 0 & 20 & -4 & 0 & -4 & -1 & 0 \\ -1 & -4 & -1 & -4 & 20 & -4 & -1 & -4 & -1 \\ 0 & -1 & -4 & 0 & -4 & 20 & 0 & -1 & -4 \\ 0 & 0 & 0 & -4 & -1 & 0 & 20 & -4 & 0 \\ 0 & 0 & 0 & -1 & -4 & -1 & -4 & 20 & -4 \\ 0 & 0 & 0 & 0 & -1 & -4 & 0 & -4 & 20 \end{pmatrix}$$

The matrix is 9×9, since $h = 1/4$ corresponds to $N = 3$ (i.e. 3 unknowns in each direction, or 9 unknowns total).

Question 3. (30 pts.) Fourier series of $f(x) = x$.

Let $f(x)$ be the 2π-periodic extension of the function $g(x) = x$ defined on the interval $-\pi \le x < \pi$. The function $f(x)$ is called the *sawtooth wave.*

(a) Sketch $f(x)$.

(b) Find the Fourier series for $f(x)$. Write the series in terms of sines and cosines. You can either find the sine/cosine series directly or deduce it from the complex series.

Hint: Use integration by parts.

(c) Use Parseval's theorem to deduce the following identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

SOLUTION:

(a) See the plot in Figure 3.

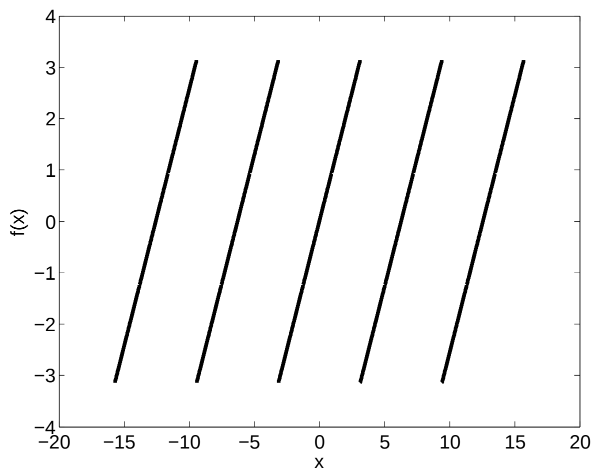

FIGURE 3. Solution to Question 3a: plot of the sawtooth wave.

(b) We use integration by parts to find the Fourier series of $f(x)$. Note that $a_n = 0$ for all n, since $g(x) = x$ is an odd function of x.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, \mathrm{d}x \\ &= -\frac{1}{\pi n} x \cos nx \Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx \, \mathrm{d}x \\ &= -\frac{1}{\pi n}\left(\pi \cos \pi n + \pi \cos(-n\pi)\right) + \frac{1}{n^2} \sin nx|_{-\pi}^{\pi} \\ &= -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n} \end{aligned}$$

So the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$
$$= 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \frac{2}{5}\sin 5x + \dots$$

(c) Parseval's theorem states that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,\mathrm{d}x = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

The left-hand side is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x = \frac{1}{2\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi}$$
$$= \frac{\pi^2}{3}$$

The right-hand side is (since $a_n = 0$)

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2 = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Equating the two sides, we obtain the desired identity

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Question 4. (20 pts.) Fourier series of $f(x) = \mathrm{e}^{-x}$.

Let $f(x)$ be the 2π-periodic extension of the function $g(x) = \mathrm{e}^{-x}$ defined on the interval $-\pi \le x < \pi$.

(a) Sketch $f(x)$.

(b) Find the complex Fourier series for $f(x)$.

SOLUTION:

(a) See the plot in Figure 4.

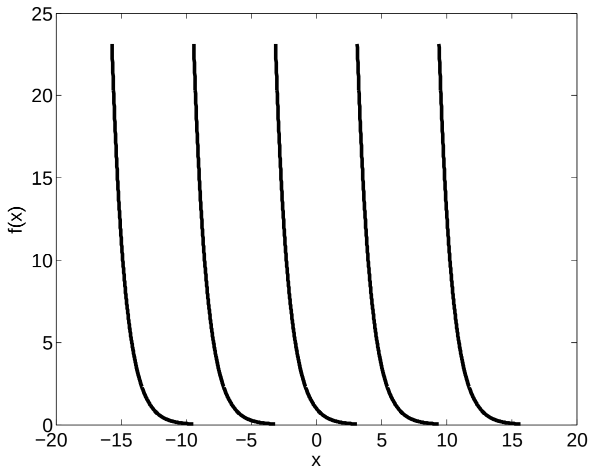

FIGURE 4. Solution to Question 4a: plot of the 2π-periodic extension of $g(x) = \mathrm{e}^{-x}$.

(b) The complex Fourier series for $f(x)$ has the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-inx} \, \mathrm{d}x$$

The Fourier coefficients are thus

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-(1+in)x} \, \mathrm{d}x \\ &= -\frac{1}{2\pi\,(1+in)} \, \mathrm{e}^{-(1+in)x} \Big|_{x=-\pi}^{x=\pi} \\ &= -\frac{1}{2\pi\,(1+in)} \left(\mathrm{e}^{-\pi} \mathrm{e}^{in\pi} - \mathrm{e}^{\pi} \mathrm{e}^{-in\pi} \right) \\ &= -\frac{(-1)^n}{2\pi(1+in)} \left(\mathrm{e}^{-\pi} - \mathrm{e}^{\pi} \right) \quad \text{since } \mathrm{e}^{i\pi} = -1 \\ &= \frac{(-1)^n \sinh \pi}{\pi(1+in)} \quad \text{where } \sinh x = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2} \end{aligned}$$

We thus obtain the Fourier series

$$f(x) = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+in} \mathrm{e}^{inx}$$

Bonus (5 pts.). Fourier coefficients and differentiability

(a) Let $f_2(x)$ be the periodic extension of the function $g_2(x) = x^2$ defined on the interval $-1 \le x < 1$. Show that $f_2(x)$ is continuous, but that f_2' is not continuous.

(b) Let $f_3(x)$ be the periodic extension of the function $g_3(x) = x^3 + ax^2 + bx$ defined on the same interval. Find the constants a and b such that f_3 and f_3' are both continuous. Show that f_3'' is not continuous.

(c) Let $f_4(x)$ be the periodic extension of the function $g_4(x) = x^4 + ax^3 + bx^2 + cx$ defined on the same interval. Find the constants a, b and c such that f_4, f_4' and f_4'' are continuous. Show that f_4''' is not continuous.

(d) Find the Fourier coefficients of f_2, f_3 and f_4 in Matlab. Show plots demonstrating that they decay (in absolute value) like $1/n^2$, $1/n^3$ and $1/n^4$, respectively. Explain why this is the case.

(e) Let $f(x) = \mathrm{e}^{\cos(\pi x)}$. Find the Fourier coefficients of f in Matlab, and show a plot demonstrating that they decay exponentially in n. Explain why this is the case.

Hints for parts (d) and (e):

- You can find the integral of a function (for example, $f(x) = x^2$) on the interval $[a, b]$ in Matlab using the following command:

  ```
  quadgk(@(x) x.^2,a,b)
  ```

  Note the dot after x.

- To demonstrate the rate of decay of the Fourier coefficients, you may want to think about plotting their log ...

SOLUTION: In parts (a)–(c), since each of the functions $g_i(x)$ is continuously differentiable on the interval $[-1, 1]$, we only need to check the continuity of the derivatives of f_i (their periodic extensions) at the endpoints $x = -1$ and $x = 1$.

(a) It is clear that $g_2(1) = g_2(-1) = 1$, so f_2 is continuous. Since $g_2'(x) = 2x$, we see that $g_2'(-1) \neq g_2'(1)$, so f_2' is not continuous.

(b) We want to construct $g_3(x)$ such that $g_3(1) = g_3(-1)$ and $g_3'(-1) = g_3'(1)$, which ensures that both f_3 and f_3' are continuous. These conditions yield the algebraic equations

$$1 + a + b = -1 + a - b$$
$$3 + 2a + b = 3 - 2a + b$$

The solution to this system is $a = 0$ and $b = -1$. The function is thus $g_3(x) = x^3 - x$. Note that $g_3''(x) = 6x$, so $g_3''(-1) \neq g_3''(1)$, implying that f_3'' is not continuous.

(c) We want to construct $g_4(x)$ such that $g_4(1) = g_4(-1)$, $g_4'(-1) = g_4'(1)$ and $g_4''(1) = g_4''(-1)$. We thus obtain the algebraic equations

$$1 + a + b + c = 1 - a + b - c$$
$$4 + 3a + 2b + c = -4 + 3a - 2b + c$$
$$12 + 6a + 2b = 12 - 6a + 2b$$

The solution to this system of equations is $a = c = 0$ and $b = -2$. The function is thus $g_4(x) = x^4 - 2x$. Note that $g_4''' = 24x$, so $g_4'''(1) \neq g_4'''(-1)$, implying that f_4''' is not continuous.

(d) Note that $g_2(x)$ is an even function, so the sine coefficients b_i will be zero. The cosine coefficients have the formula

$$a_n = \int_{-1}^{1} x^2 \cos(n\pi x)\,\mathrm{d}x$$

We use the following Matlab code to compute the cosine coefficients:

```
nv = 1:100;
g2 = @(x) x.^2;
cv = zeros(1,length(nv));
for ind = 1:length(nv)
    n = nv(ind);
    cv(ind) = quadgk(@(x) g2(x).*cos(pi*n*x),-1,1);
end;
plot(log(nv),log(abs(cv)),'-k','LineWidth',3)
```

The output is shown in Figure 5. Note that we plot on a log-log scale: since we expect $|a_n| = O(n^{-2})$ as $n \to \infty$, plotting $\log n$ against $\log |a_n|$ should yield a line of slope -2, as shown in the figure. We do the same for f_3 and f_4, except we plot the sine coefficients for f_3 since it is an odd function. Since the lines in Figure 5 have slopes -2, -3, and -4, respectively, we thus confirm that the Fourier coefficients decay like $O(n^{-2})$, $O(n^{-3})$ and $O(n^{-4})$, respectively. This is because the differentiability of a function is related to the rate of decay of its Fourier coefficients. That is, if $f(x)$ has m continuous derivatives, its Fourier coefficients decay like $O(n^{-(m+2)})$ as $n \to \infty$.

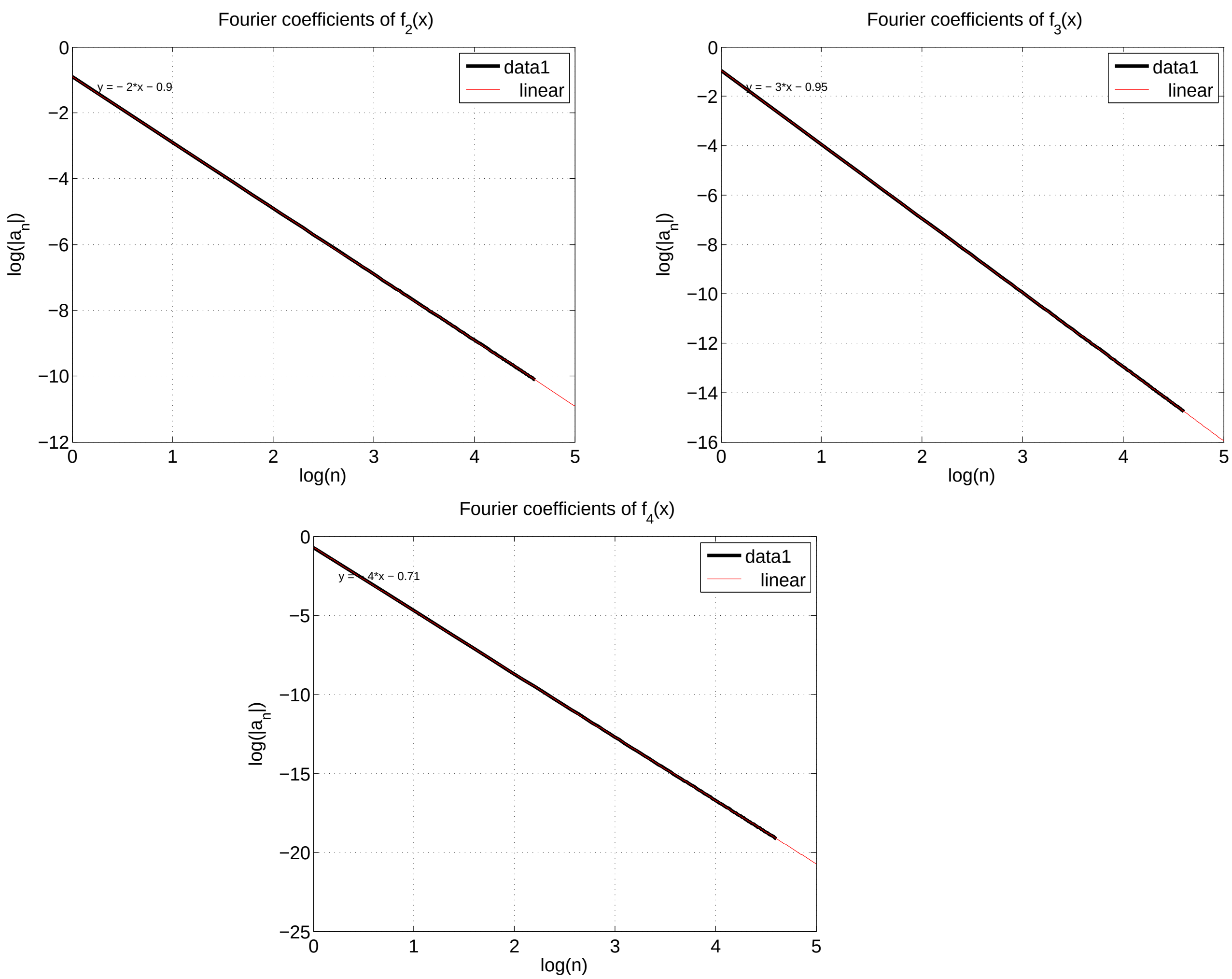

FIGURE 5. Solution to Question 4d.

(e) We find the cosine coefficients of $f(x) = \mathrm{e}^{\cos(\pi x)}$ using the same code as in part (d). In Figure 6, we plot n (rather than $\log n$) against $\log |a_n|$ and observe a straight line, confirming that the Fourier coefficients decay exponentially. Note that the line becomes flat below $-36 \approx \log(10^{-15})$, which is machine precision. The exponential decay of the Fourier coefficients, as opposed to the polynomial decay observed in part (d), reflects the fact that $f(x)$ is an infinitely differentiable function of x.

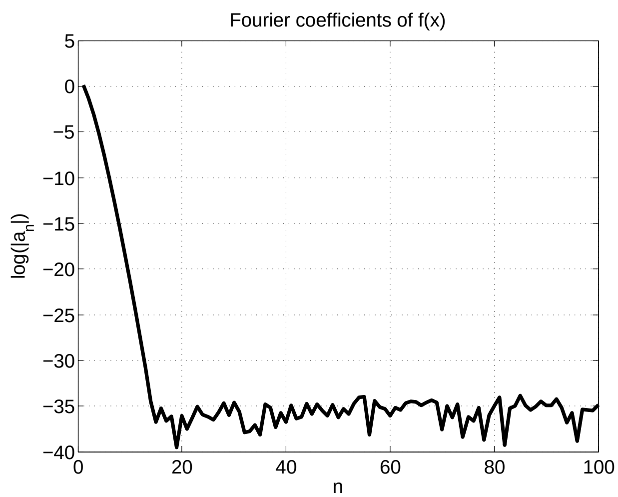

FIGURE 6. Solution to Question 4e.