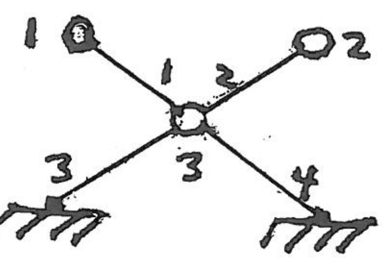
18.085 Exam 2 November 9, 2011 Professor Strang

Your PRINTED name is: SOLUTIONS

Your class number is

1.
2.
3.
4.



1. (a) (5 points) A truss is in the shape of an X with 45° angles and 2 fixed nodes at the bottom. Based on the number m of bars and n of unknown displacements, how many independent solutions do you expect to $Au = 0$?

$m=4$ $n=6$ (2 at each of 3 nodes)

We expect $6-4=2$ Solutions

(b) (10 points) You can answer without writing down the matrix A. Give the components $u_1^H, u_1^V, \ldots, u_3^V$ for a full set of independent solutions of $Au = 0$. DRAW THESE MECHANISMS!

$\boxed{1}$ Bar 1 turns $u=(1,1,0,0,0,0)$

$\boxed{2}$ Bars 1 and 2 turn $u=(1,1,1,-1,0,0)$

OR JUST Bar 2 turns by itself $u=(0,0,1,-1,0,0)$

(c) (10 points) What is row 1 of the matrix A (corresponding to upper left bar 1)?

row 1 (from bar 1) $= \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

(I hope correct signs)

2. (a) (5 points) Explain what condition the components $(v_1, v_2) = (\partial u/\partial x, \partial u/\partial y)$ of a gradient field must satisfy, and give the reason why.

$$\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x} \quad \text{because} \quad \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y}$$

(b) (10 points) Find a potential $u(x, y)$ if $v(x, y) = (1, 2) =$ constant velocity.

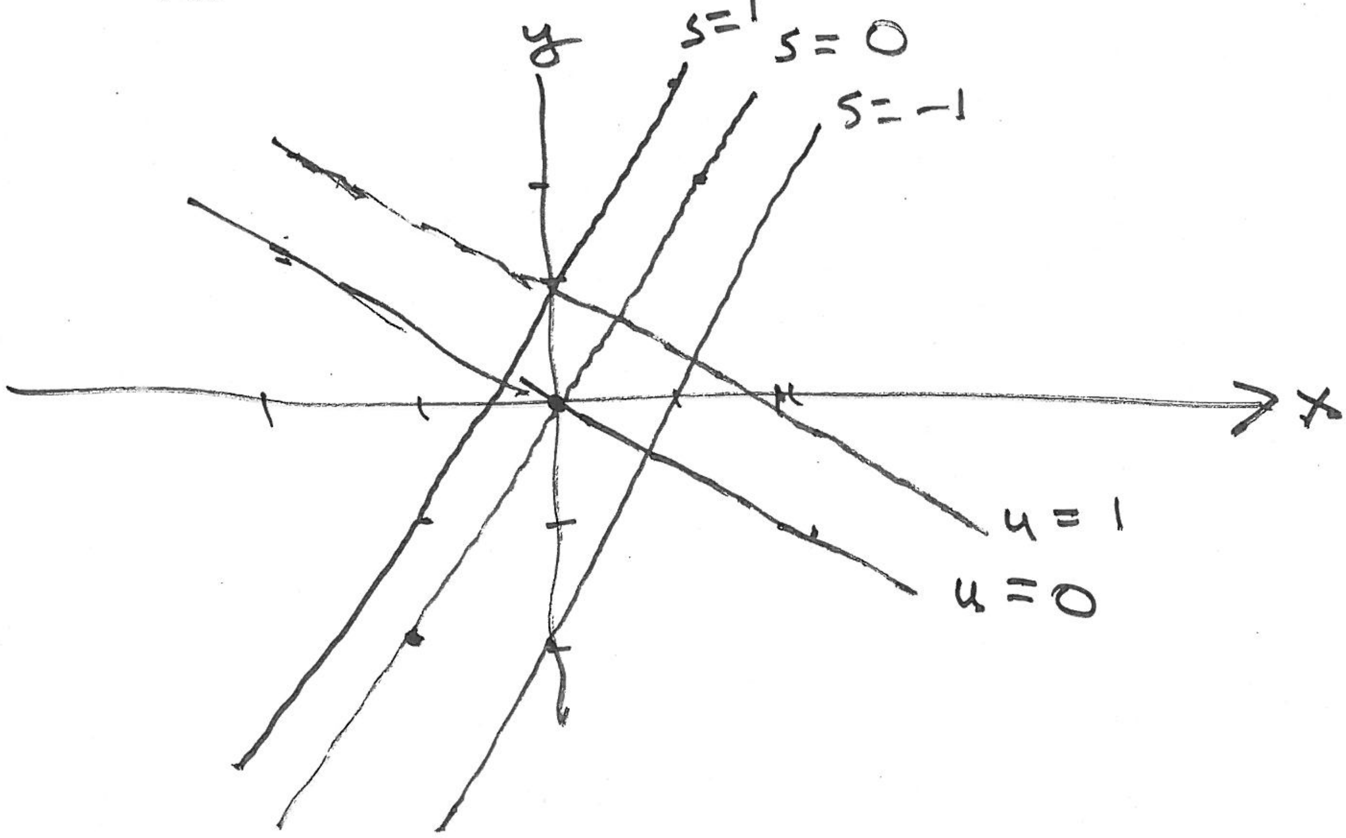
$u = x + 2y$

Using this $u(x, y)$, find a stream function $s(x, y)$ so that the Cauchy-Riemann equations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x}.$$

$\frac{\partial s}{\partial y} = 1 \quad -\frac{\partial s}{\partial x} = 2 \quad$ so $s = -2x + y$ (+ const)

(c) (5 points) Draw a few equipotential curves $u(x, y) =$ constant and a few streamlines $s(x, y) =$ constant.



$f =$

3. (a) (5 points) In class I never checked that $(x + iy)^n$ solves Laplace's equation. Please substitute it into the equation to confirm. [Then its real and imaginary parts also work.]

$$\frac{df}{dx} = n(x+iy)^{n-1} \qquad \frac{d^2f}{dx^2} = n(n-1)(x+iy)^{n-2}$$

$$\frac{df}{dy} = in(x+iy)^{n-1} \qquad \frac{d^2f}{dy^2} = i^2 n(n-1)(x+iy)^{n-2}$$

(b) (10 points) Find the real and imaginary parts u and s of the function $1/z = 1/(x+iy)$. Give the answers u and s in x, y coordinates and also in polar r, θ coordinates.

$$\frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = \frac{\cos\theta}{r} - i\,\frac{\sin\theta}{r}$$

(c) (10 points) Solve Laplace's equation **outside the unit circle** $r^2 = x^2 + y^2 = 1$ if the boundary condition is $u = u_0 = y$ on the circle.

The solution that goes to 0 as $r \to \infty$ is $u = \frac{\sin\theta}{r}$

* The answer $u = r\sin\theta$ does satisfy the BC.
-2 points only

4. (a) (10 points) Find the weak form of the 1-D equation

Free - fixed

$$-\frac{d}{dx}\left(\frac{du}{dx}\right) = \delta\left(x - \frac{1}{2}\right)$$

with $du/dx(0) = 0$ and $u(1) = 0$. You must tell me what boundary conditions $u(x)$ and $v(x)$ are required to satisfy in the weak form.

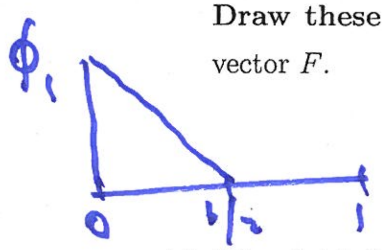
$$\int_0^1 \frac{du}{dx}\frac{dv}{dx}\,dx = \int_0^1 f(x)\,v(x)\,dx$$ with $u(1) = v(1) = 0$

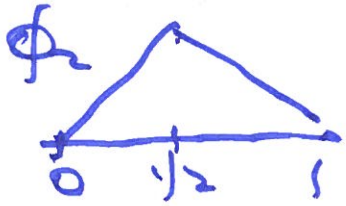
$\rightarrow$ No BC at $x=0$ on u, v

(b) (10 points) Suppose $h = \frac{1}{2}$ and we use two continuous piecewise linear functions (hat-type functions) ϕ_1 and ϕ_2:

trial functions ϕ_i = test functions V_i = hat-type functions $= 1$ at one node.

Draw these functions. Find the 2 by 2 stiffness matrix K and the 2-component vector F.





$$\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) (10 points) Solve $KU = F$ to find the finite element solution $U = (U_1, U_2)$ at the nodes. DRAW the graph of this solution $U(x) = U_1\phi_1 + U_2\phi_2$.

$$U = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

P.S. This is the exactly correct solution $u(x)$ to the differential equation.
(Always lucky in 18.085.)

height $\rightarrow$ 1/2

−4 for wrong height