18.085 Quiz 3 December 8, 2006 Professor Strang

Your PRINTED name is: **SOLUTIONS**

Grading
1
2
3

1) **(30 pts.)** (a) Suppose $f(x)$ is a *periodic* function:

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ e^{-x} & \text{for } 0 \le x \le \pi \\ f(x + 2\pi n) & \text{for every integer } n \end{cases}$$

Find the coefficients c_k in the complex Fourier series $f(x) = \sum c_k e^{ikx}$. What is c_0? What is $\sum_{-\infty}^{\infty} |c_k|^2$?

(b) Draw a graph of $f(x)$ from -2π to 2π. Also draw a careful graph of df/dx. How quickly do the coefficients of $f(x)$ decay as $k \to \infty$ and why?

(c) Find the Fourier coefficients d_k of df/dx. Do they approach a constant (or what pattern do they approach) as $k \to \infty$? Explain the pattern from your graphs.

Solution.

(a) $c_k = \dfrac{1}{2\pi}\displaystyle\int_0^{\pi} e^{-x}e^{-ikx}\,dx = \dfrac{1}{2\pi}\dfrac{e^{-(1+ik)x}}{-(1+ik)}\Bigg|_0^{\pi} = \dfrac{1}{2\pi}\dfrac{1-e^{-(1+ik)\pi}}{1+ik} = \dfrac{1-(-1)^k e^{-\pi}}{2\pi(1+ik)}$

$c_0 = \dfrac{1-e^{-\pi}}{2\pi} \qquad \sum |c_k|^2 = \dfrac{1}{2\pi}\displaystyle\int_0^{\pi}(e^{-x})^2\,dx = \dfrac{1-e^{-2\pi}}{4\pi}$

(b) The graph of $f(x)$ includes a jump of 1 at $x=0$ and a drop of $e^{-\pi}$ at $x=\pi$. So df/dx includes $\delta(x) - e^{-\pi}\,\delta(x-\pi)$. (Both function have e^{-x} from 0 to π.)

The coefficients of $f(x)$ decay like $1/k$ because of the two jumps.

(c) The coefficients of df/dx are

$$d_k = ik\,c_k = \frac{ik}{2\pi(1+ik)}(1+(-1)^k e^{-\pi}).$$

As $k \to \infty$ they do not approach a constant (which would be 1, coming from $\delta(x)$). Instead the limiting pattern alternates between $1+e^{-\pi}$ and $1-e^{-\pi}$, because $f(x)$ has *two jumps*.

2) **(33 pts.)** (a) Can you complete this 4-step MATLAB code to compute the cyclic convolution $f \circledast g = h$? I suggest `fhat`, `ghat`, `hhat` for their transforms.

```
1.  fhat = fft(f)
2.  ghat = fft(g)
3.  hhat = fhat .* ghat
4.  h = ifft(hhat)
```

(It is equally possible to start with the inverse discrete transform `ifft`. The only difference will be a factor of N somewhere, which I forgive! If you don't know MATLAB notation for commands 2, 3, 4 you can use words. MATLAB's `fft(f)` and `ifft(f)` automatically determine the length of f.)

(b) Suppose each of your quiz grades is a random variable (don't know how I thought of this). The probability of grade j on each quiz ($j = 0, \ldots, 100$) is p_j. The "generating function" for that quiz is $P(z) = \sum p_j z^j$. What is the probability s_k that the sum of your grades on 2 quizzes is k? Give a nice formula for $S(z) = \sum s_k z^k$.

(c) The chance of grade $j = (70, 80, 90, 100)$ on one quiz is $p = (.3, .4, .2, .1)$. What is the expected value (mean m) for the grade on that quiz?

Show that this quiz average m agrees with dP/dz at $z = 1$. What are the probabilities s_k for the sum of two grades? Give numbers or a MATLAB code for the s_k.

Solution.

(b) The two grades are i and j with probability $p_i p_j$. Looking at all pairs that add to k,

$$s_k = \sum_{i+j=k} p_i p_j = \sum p_i p_{k-i} \quad \text{and} \quad s = p * p\,.$$

The convolution rule (multiplying polynomials is convolution of coefficients) says that $S(z) = (P(z))^2$.

* I should have worded this problem more clearly.*

(c) The expected value (the mean m) is

$$(.3)(70) + (.4)(80) + (.2)(90) + (.1)(100) = 81\,.$$

This is the derivative at $z = 1$ of

$$P(z) = (.3)z^{70} + (.4)z^{80} + (.2)z^{90} + (.1)z^{100}$$

For the probabilities s_k, part (b) says that we have to convolve $p * p$. Noncyclic convolution is `conv(p,p)` — or pad p by extra zeros and use the cyclic code in part (a) — or compute $(3421)^2$ without carrying:

$$\begin{array}{ccccccc}
 & & & 3 & 4 & 2 & 1 \\
 & & & 3 & 4 & 2 & 1 \\
\cline{4-7}
 & & & 3 & 4 & 2 & 1 \\
 & & 6 & 8 & 4 & 2 & \\
 & 12 & 16 & 8 & 4 & & \\
9 & 12 & 6 & 3 & & & \\
\hline
9 & 24 & 28 & 22 & 12 & 4 & 1
\end{array} = \text{percentages adding to } 100$$

3) **(37 pts.)**

(a) The hat function $H(x) = 1-|x|$ for $-1 \le x \le 1$ has height 1 and area 1 and integral transform $\widehat{H}(k) = (2 - 2\cos k)/k^2$. Find the transform $\widehat{R}(k)$ of the roof function $R(x)$:

$$R(x) = \textbf{box} + \textbf{hat} = 2 - |x| \quad \text{for } -1 \le x \le 1, \;\; 0 \text{ else.}$$

(b) What is the value of $\widehat{R}(k)$ at $k = 0$ and how does this connect to the graph of the roof?

(c) Suppose $R(x)$ is the response of a sensor to a point source $\delta(x)$ at $x = 0$. The sensor is shift-invariant (shifted response when source is shifted). The output F from a distributed source $U(x)$ is the convolution $F = R * U$. Describe how to find $U(x)$ if you know $F(x)$.

(d) There could be a difficulty with your solution method in part (c). That would arise if ______ $= 0$. For 1 point, does this difficulty appear in this example?

Solution.

(a) The box on $[-1,1]$ has transform $(e^{ik} - e^{-ik})/ik = 2\sin k/k$. Then $R = \textbf{box} + \textbf{hat}$ has

$$\widehat{R} = \widehat{\textbf{box}} + \widehat{\textbf{hat}} = \frac{2\sin k}{k} + \frac{2 - 2\cos k}{k^2}$$

Note: The $1/k$ decay rate comes from the jumps in the box function. The $1/k^2$ terms come from corners in the hat.

(b) $\widehat{R}(0) = 3$ because the area under $R(x)$ is $\int_{-1}^{1} R(x)\, e^{0x}\, dx = 3$.

(c) Take transforms of $F = R * U$ to find $\widehat{F} = \widehat{R}\,\widehat{U}$. Then $\widehat{U} = \widehat{F}/\widehat{R}$. Invert this transform to find $U(x)$.

(d) There is a difficulty if $\widehat{R}(k) = 0$ for any frequencies k. This does appear in the example when $k = 2\pi, 4\pi, \ldots$