18.085 Quiz 3 December 9, 2005 Professor Strang

Your PRINTED name is: SOLUTIONS **Grading** **1** **2** **3**

1) **(30 pts.)** (a) Solve this *cyclic convolution* equation for the vector d. (I would transform convolution to multiplication.) Notice that $c = (5,0,0,0) - (1,1,1,1)$. The equation is like deconvolution.

$$c \circledast d = (4,-1,-1,-1) \circledast (d_0, d_1, d_2, d_3) = (1,0,0,0)\,.$$

(b) Why is there no solution d if I change c to $C = (3,-1,-1,-1)$? Try it. Can you find a nonzero D so that $C \circledast D = (0,0,0,0)$?

Solution.

(a) Here $n = 4$. The transform Fc is $5(1,1,1,1)-(4,0,0,0) = (1,5,5,5)$. The right side has transform $(1,1,1,1)$. Multiplication (or division!) gives $(1,.2,.2,.2) = .8(1,0,0,0) + .2(1,1,1,1)$ which comes from $.8(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}) + .2(1,0,0,0) = (.4,.2,.2,.2) = d$.

(b) The transform FC is $4(1,1,1,1) - (4,0,0,0) = (0,4,4,4)$. We can't divide by zero! The vector $D = (1,1,1,1)$ solves $C * D = (0,0,0,0)$.

Note for the future. Express the same problem with circulant matrices:

(a) $$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} .4 & .2 & .2 & .2 \\ .2 & .4 & .2 & .2 \\ .2 & .2 & .4 & .2 \\ .2 & .2 & .2 & .4 \end{bmatrix} = I$$

(b) No solution when the matrix is singular—a zero eigenvalue! (The eigenvalues are the discrete transforms!!)

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2) **(36 pts.)** (a) If $f(x) = e^{-x}$ for $0 \le x \le 2\pi$, extended periodically, find its (complex) Fourier coefficients c_k.

(b) What is the decay rate of those c_k and how could you see the decay rate from the function $f(x)$?

(c) Compute $\sum_{-\infty}^{\infty} |c_k|^2$ for those c's as an ordinary number. [1 point question: How in the world could you find $\sum_{-\infty}^{\infty} |c_k|^4$? Don't try!]

(d) Solve this periodic differential equation to find $u(x)$:

$$u'(x) + u(x) = \delta(x) + \text{ periodic train of deltas}$$

Solution.

(a) $c_k = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{-x}\, e^{-ikx}\, dx = \dfrac{e^{-(1+ik)x}}{-2\pi(1+ik)}\Big|_0^{2\pi} = \dfrac{1-e^{-(1+ik)2\pi}}{2\pi(1+ik)} = \dfrac{1-e^{-2\pi}}{2\pi(1+ik)}$

(b) Decay rate $\frac{1}{k}$ because $f(x)$ jumps from $e^{-2\pi}$ to 1 at the end of every 2π period.

(c) $\displaystyle\sum_{-\infty}^{\infty} |c_k|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-2x}\, dx = \frac{1-e^{-4\pi}}{4\pi}$

To find $\sum |c_k|^4$ we want the function $F(x)$ whose Fourier coefficients are c_k^2. By the convolution rule $F(x) \approx f * f$ (which is painfully computable since e^{-x} is easy to integrate).

(d) $(ik+1)c_k = \dfrac{1}{2\pi}$ so $c_k = \dfrac{1}{2\pi(1+ik)}$, which agrees with part (a) after dividing by the constant: $u(x) = \dfrac{f(x)}{1-e^{-2\pi}} = \displaystyle\sum \frac{e^{ikx}}{2\pi(1+ik)}$.

3) **(34 pts.)** Suppose $f(x)$ is a *half-hat function* $(-\infty < x < \infty)$.

$$f(x) = \begin{cases} 1 - x & \text{for } 0 \le x \le 1 \\ 0 & \text{for all other } x \end{cases}$$

(a) Draw a graph of $f(x)$ on the whole line $-\infty < x < \infty$ and ALSO a graph of its derivative $g(x) = df/dx$.

(b) What is the transform (Fourier integral) $\widehat{g}(k)$ of df/dx?

(c) What is the transform $\widehat{f}(k)$ of $f(x)$? Does it have the decay rate you expect? What is $\widehat{f}(0)$?

(d) Christmas present: Is the convolution $f(x) * f(x)$ of the half-hat with itself equal to the usual full hat $H(x)$? (*Yes or no answer*, 4 points).

THANK YOU FOR TAKING 18.085! 18.086 will be good small projects in scientific computing.

Solution.

(a) $g(x) = \delta(x) -$ unit square wave on $[0, 1]$

(b) $\widehat{g}(x) = 1 - \dfrac{1 - e^{-ik}}{ik} = \dfrac{ik - 1 + e^{-ik}}{(ik^2)}$

(c) $\widehat{f}(k) = \dfrac{\widehat{g}(k)}{ik} = \dfrac{ik - 1 + e^{-ik}}{(ik)^2} = \dfrac{ik - 1 + (1 - ik - k^2/2 \cdots)}{(ik)^2}$

$= \left(\frac{1}{2} \text{ at } k = 0\right) =$ area under $f(x)$!

Decay rate $\dfrac{1}{k}$ because $f(x)$ has a step at $x = 0$.

(d) No way.