18.085 Quiz 3 December 2, 2004 Professor Strang

Your name is: ______________________________

Grading 1.
2.
3.

Thank you for taking 18.085 ! I hope to see you in 18.086 !!

1) **(40 pts.)** This question is about 2π-*periodic functions.*

(a) Suppose $f(x) = \sum c_k e^{ikx}$ and $g(x) = \sum d_l e^{ilx}$. Substitute for f and g and integrate to find the coefficients q_n in this convolution:

$$h(x) = \int_0^{2\pi} f(t)\, g(x-t)\, dt = \int_0^{2\pi} f(x-t)\, g(t)\, dt = \sum q_n e^{inx}\,.$$

(b) Compute the coefficients c_k for the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{for } 1 \le x \le 2\pi \end{cases}$$

What is the *decay rate* of these c_k ? What is $\sum |c_k|^2$?

(c) Keep that $f(x)$ in parts (c)–(d). If $g(x)$ also has a jump, will the convolution $h(x)$ have a jump ? Compare the decay rates of the d's and q's to find the behavior of $h(x)$: delta function, jump, corner, or what ?

(d) Find the derivative dh/dx at $x = 0$ in terms of two values of $g(x)$. (You could take the x derivative in the convolution integral.)

x

2) **(30 pts.)** (a) We want to compute the *cyclic convolution* of $f = (1, 0, 1, 0)$ and $g = (1, 0, -1, 0)$ in two ways. First compute $f *_C g$ directly—either the formula at the end of p. 294 or from $1 + w^2$ and $1 - w^2$.

(b) Now compute the discrete transforms c (from f) and d (from g). Then use the convolution rule to find $f *_C g$.

(c) I notice that the usual dot product $\overline{f}^{\mathrm{T}} g$ is zero. Maybe also $\overline{c}^{\mathrm{T}} d$ is zero. Question for any c and d:

$$\text{If} \quad \overline{c}^{\mathrm{T}} d = 0 \quad \text{deduce that} \quad \overline{f}^{\mathrm{T}} g = 0.$$

xx

3) **(30 pts.)** This question uses the Fourier integral to study

$$-\frac{d^2u}{dx^2} + u(x) = \begin{cases} 1 & \text{for } -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

(a) Take Fourier transforms of both sides to find a formula for $\widehat{u}(k)$.

(b) What is the decay rate of this $\widehat{u}$? At what points x is the solution $u(x)$ not totally smooth? Describe $u(x)$ at those points: delta, jump in $u(x)$, jump in du/dx, jump in d^2u/dx^2, or what?

(c) We know that the Green's function for this equation (when the right side is $\delta(x)$) is

$$G(x) = \tfrac{1}{2}e^{-|x|} = \begin{cases} \tfrac{1}{2}e^{-x} & \text{for } x \ge 0 \\ \tfrac{1}{2}e^{x} & \text{for } x \le 0 \end{cases}$$

Find the solution $u(x)$ at the particular point $x = 2$.

xxx