18.085 **Quiz 2** Professor Strang

Name______________ November 4, 2002

Grading 1
2
3
– – – – – –

Problem 1 (33 points)

This question is about a fixed-free hanging bar (made of 2 materials) with a point load at $x = \frac{3}{4}$:

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = \delta\left(x - \frac{3}{4}\right)$$
$$u(0) = 0$$
$$w(1) = 0$$

Suppose that

$$c(x) = \begin{cases} 1, & x < \frac{1}{2} \\ 4, & x > \frac{1}{2} \end{cases}$$

a) Which of u, $\frac{du}{dx}$, and $w = c\frac{du}{dx}$ have jumps at (i) $x = \frac{1}{2}$ and (ii) $x = \frac{3}{4}$?

b) Solve for $w(x)$ and draw its graph from $x = 0$ to $x = 1$.

c) Solve for $u(x)$ and draw its graph from $x = 0$ to $x = 1$.

Problem 2 (34 points)

a)

(i) Find the real part $u(x,y)$ and the imaginary part $s(x,y)$ of

$$f(z) = \frac{1}{z} = \frac{1}{x+iy}$$

(ii) Also find $u(r,\theta)$ and $s(r,\theta)$ for the same function expressed in polar coordinates:

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}}$$

b) Draw the equipotential curve $u(x,y) = \frac{1}{2}$ and the streamline $s(x,y) = \frac{1}{2}$. (I suggest to use x-y coordinates and "clear out" denominators.) What shapes are these two curves?

c) What can you say about $u(x,y)$ (what condition does it satisfy) along the line $s = \frac{1}{2}$?

Problem 3 (33 points)

a). Suppose that the Laplacian of $F(x,y)$ is zero:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0.$$

Show that $u = \frac{\partial F}{\partial y}$ and $s = \frac{\partial F}{\partial x}$ satisfy the Cauchy-Riemann equations.

b). Which of these vector fields are gradients of some function $u(x,y)$ and what is that function? Does $u(x,y)$ solve Laplace's equation $\text{div}(\text{grad}\, u) = 0$?

(i) $v(x,y) = (x^2, y^2)$

(ii) $v(x,y) = (y^2, x^2)$

(iii) $v(x,y) = (x+y, x-y)$

c) (i) Find the solution to Laplace's equation inside the unit circle $r^2 = x^2 + y^2 = 1$ if the boundary condition on the circle is $u = u_0(\theta) = \frac{1}{2} + \cos\theta + \cos 2\theta$. (OK to use polar coordinates.)
(ii) Find the numerical value of the solution u at at the center and at the point $x = \frac{1}{2}, y = 0$.