18.085 QUIZ 2 SOLUTIONS

PROBLEM 1

(a)

Solution

$$w(x) = \begin{cases} 1; & x < \frac{1}{3} \\ 0; & x > \frac{1}{3} \end{cases}$$

(b)

We need to separate the problem in 3 pieces

$$A \;:\; \left(0; \frac{1}{3}\right)$$
$$B \;:\; \left(\frac{1}{3}; \frac{2}{3}\right)$$
$$C \;:\; \left(\frac{2}{3}; 1\right)$$

For each of the pieces, the equation $w(x) = c(x)\frac{du(x)}{dx}$ becomes

$$\begin{array}{ccc} A & B & C \\ \frac{du}{dx} = \frac{1}{6} & \frac{du}{dx} = \frac{1}{12} & \frac{du}{dx} = 0 \end{array}$$

and the solutions are

$$\begin{array}{ccc} A & B & C \\ u(x) = \frac{1}{6}x + C_1 & u(x) = \frac{1}{12}x + C_2 & u(x) = C_3 \end{array}$$

The left boundary condition $u(0) = 0$ yields $C_1 = 0$. Continuity of $u(x)$ at $x = \frac{1}{3}$ can be expressed as

$$\frac{1}{6} * \frac{1}{3} = \frac{1}{12} * \frac{1}{3} + C_2$$

and so $C_2 = \frac{1}{36}$. Continuity at $x = \frac{2}{3}$ yields $C_3 = \frac{1}{12} * \frac{2}{3} + \frac{1}{36} = \frac{1}{12}$ and so the final answer is

$$u(x) = \begin{cases} \frac{1}{6}x; & 0 < x < \frac{1}{3} \\ \frac{1}{12}x + \frac{1}{36}; & \frac{1}{3} < x < \frac{2}{3} \\ \frac{1}{12}; & \frac{2}{3} < x < 1 \end{cases}$$

(c)

(i)

The function $w(x)$ is the response of the material to the force. Whatever response is necessary it will be created by a body in equilibrium. And so, unless $f(x)$ has a $\delta(x)$ component, $w(x)$ is continuous. However, $\frac{du}{dx}$ depends on the material properties and will experience a jump if $c(x)$ jumps. It will jump in such a way as to make the product $c(x)\frac{du}{dx}$ continuous.

(ii)

If $f(x)$ has a jump then, as before, $w(x)$ is continuous, and since $c(x)$ is continuous, $\frac{du}{dx}$ is continuous. If you assume that $c(x)$ has a jump then, as above, $\frac{du}{dx}$ must have a jump also.

PROBLEM 2

(a)

We have

$$\begin{aligned}\ln\left((x+iy)^2\right) &= \ln\left(\left(x^2+y^2\right)e^{i2\theta}\right) = \\ &= \ln\left(x^2+y^2\right) + i\theta\end{aligned}$$

and so the real part is

$$u(x) = \ln\left(x^2+y^2\right)$$

(b)

We have

$$\begin{aligned}\mathbf{v}(x;y) &= \nabla u(x;y) = \nabla \ln\left(x^2+y^2\right) = \\ &= \left(\frac{2x}{x^2+y^2}; \frac{2y}{x^2+y^2}\right)\end{aligned}$$

Streamlines are concentric circles around the origin.

(c)

(i) Flux $= 4\pi$ since the stream function changes by 4π when you go around the circle.

(ii) We have for the normal

$$\mathbf{n}(x;y) = \left(\frac{x}{\sqrt{x^2+y^2}}; \frac{y}{\sqrt{x^2+y^2}}\right)$$

and so

$$\mathbf{v}\cdot\mathbf{n} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)\cdot\left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right) = $$
$$= 2\frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{2}{r}$$

Multiplying this by the length ($2\pi r$) we get once again: Flux = 4π.

(d)

There's really not much of an "or". You should use the divergence theorem. Since

$$\nabla\cdot\mathbf{v} = 0$$

the flow through "square minus circle" is 0. Therefore, the flow through the square and the circle are the same. And so

$$Flux_{square} = 4\pi$$

One is tempted to use the following (incorrect) argument. Since $\nabla\cdot\mathbf{v} = 0$ the flow through the unit circle (or any shape, for that matter) must be 0. However, it is critical for the divergence theorem that there are no "bad" points inside the domain. And for $\ln r$, the point $(0,0)$ is a bad point. Strictly speaking, at the point $(0,0)$ the Laplacian of $\ln r$ does not exist. Less strictly speaking, it is a 2π multiple of the delta function

$$\nabla\cdot\left(\nabla\ln r\right) = 2\pi\delta(x,y) \tag{1}$$

and that is why for any domain Ω that contains the origin we have

$$\int_\Omega \nabla\cdot\left(\nabla\ln r\right)\,dx = 2\pi$$

In fact, the calculation that you performed in class is the physicist's/engineer's proof of the remarkable relationship (1). In three dimensions the equivalent is

$$\nabla\cdot\left(\nabla\frac{1}{r}\right) = -4\pi\delta(x,y,z)$$

Problem 3

(Writing this one from memory - don't have the problem in front of me.)

(a)

Weak form

$$\int_0^1 u_x v_x \, dx = \int_0^1 f v \, dx$$

must hold for all $v(x)$ such that

$$\begin{aligned} v(0) &= 0 \\ v(1) &= 0 \end{aligned}$$

(b)

The whole system (this got extra credit) is

$$-(N+1)\begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \vdots & \vdots & \vdots & & \\ & & \vdots & \vdots & \vdots & & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \\ \\ \\ \\ F_{N-1} \\ F_N \end{bmatrix}$$

(c)

Only the first two terms of F are effected. K is unchanged.