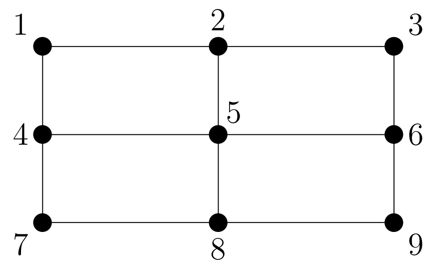
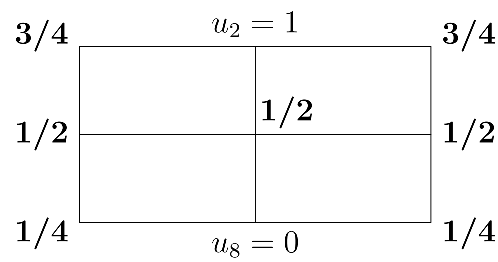
18.085 Quiz 2 November 3, 2006 Professor Strang

**Your PRINTED name is:** SOLUTIONS **Grading** **1** **2** **3** **4**

1) **(25 pts.)** This network (square grid) has 12 edges and 9 nodes.




(a) Do not write the incidence matrix! Do not give me a MATLAB code! Just tell me:

(1) How many independent columns in $A$? **8**

(2) How many independent solutions to $A^{\mathrm{T}}y = 0$? $\mathbf{12 - 8 = 4}$

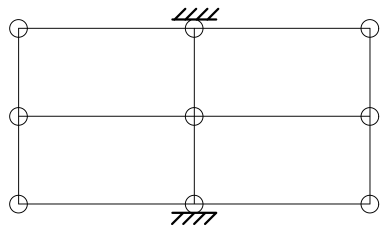
(3) What is row 5 (coming from node 5) of $A^{\mathrm{T}}A$?

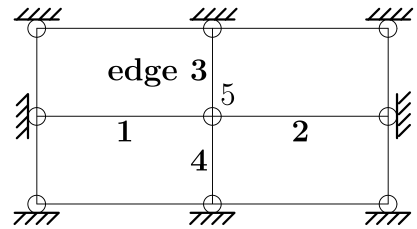
I do want the whole of row 5. $\begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{-1} & \mathbf{4} & \mathbf{-1} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \end{bmatrix}$

(b) Suppose the node 2 has voltage $u_2 = 1$, and node 8 has voltage $u_8 = 0$ (ground). All edges have the same conductance $c$. **On the second picture write** all of the other voltages $u_1$ to $u_9$. Check equation 5 of $A^{\mathrm{T}}Au = 0$ (at the middle node).

$$\mathbf{-\frac{1}{2} - \frac{1}{2} + 4\left(\frac{1}{2}\right) - 1 - 0 = 0}$$

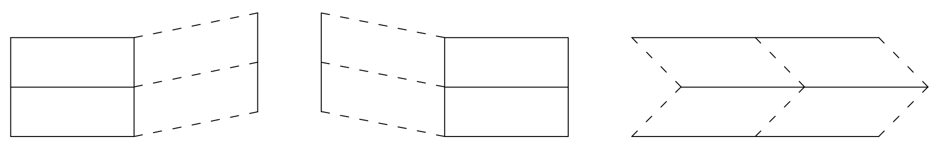
2) **(25 pts.)** Suppose that square grid becomes a plane truss (usual pin joints at the 9 nodes). Nodes 2 and 8 now have supports so $u_2^{\mathrm{H}} = u_2^{\mathrm{V}} = u_8^{\mathrm{H}} = u_8^{\mathrm{V}} = 0$.






Any numbering of edges

(a) Think about the strain-displacement matrix $A$. Are there any mechanisms that solve $Au = 0$? If there are, **tell me carefully how many** and **draw a complete set**. ***I believe* 3.**



(b) Suppose now that all 8 of the outside nodes are fixed. Only node 5 is free to move. There are forces $f_5^{\mathrm{H}}$ and $f_5^{\mathrm{V}}$ on that node. The bars connected to it (North East South West) have constants $c_N\ c_E\ c_S\ c_W$.

What is the (**reduced**) matrix $A$ for this truss on the right? What is the **reduced** matrix $A^{\mathrm{T}}CA$? What are the **displacements** $u_5^{\mathrm{H}}$ and $u_5^{\mathrm{V}}$?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$ (any row order is OK)

$$A^{\mathrm{T}}CA = \begin{bmatrix} c_E + c_W & 0 \\ 0 & c_N + c_S \end{bmatrix}$$

$$u_5^{\mathrm{H}} = f_5^{\mathrm{H}}/(c_E + c_W)$$
$$u_5^{\mathrm{V}} = f_5^{\mathrm{V}}/(c_N + c_S)$$

For 1 point, is that truss (fixed at 8 nodes) statically determinate or indeterminate? **indeterminate**

3) **(25 pts.)** This question is about the velocity field $v(x,y) = (0,x) = w(x,y)$.

(a) Check that $\operatorname{div} w = 0$ and find a stream function $s(x,y)$. **Draw** the streamlines in the $xy$ plane and show some velocity vectors.

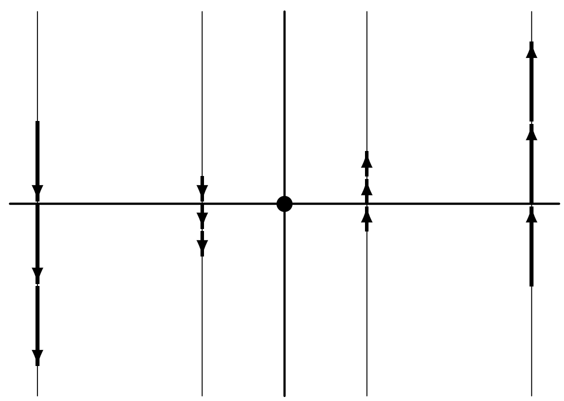
(b) Is this shear flow a gradient field ($v = \operatorname{grad} u$) or is there rotation? If you believe $u$ exists, find it. If you believe there is rotation, explain how this is possible with the streamlines you drew in part (a).

*Solution.*

(a) $\operatorname{div}(0,x) = 0 + 0 = 0$

$$0 = \frac{\partial s}{\partial y} \qquad x = -\frac{\partial s}{\partial x} \qquad \boxed{s = -\frac{1}{2}x^2}\ (+C)$$

$x =$ constant are *vertical streamlines.*



(b) Not a gradient field because $\dfrac{\partial v_1}{\partial y} = 0 \quad \dfrac{\partial v_2}{\partial x} = 1.$

Possible explanation for vorticity (rotation) $\dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y} = 1.$

Right side going faster than left side produces rotation.

4) **(25 pts.)** Suppose I use linear finite elements (hat functions $\phi(x)$ = trial functions $V(x)$). The equation has $c(x) = 1 + x$ and a point load:

**Fixed-free** $$-\frac{d}{dx}\left((1+x)\frac{du}{dx}\right) = \delta\left(x - \frac{1}{2}\right) \quad \text{with } u(0) = 0 \text{ and } u'(1) = 0\,.$$

Take $h = 1/3$ with two hats and a half-hat as in the notes.

(a) On the middle interval from 1/3 to 2/3, $U(x)$ goes **linearly** from $U_1$ to $U_2$. Compute

$$\int_{1/3}^{2/3} c(x)\big(U'(x)\big)^2\,dx \quad \text{and} \quad \int_{1/3}^{2/3} \delta\left(x - \frac{1}{2}\right) U(x)\,dx\,.$$

Write those answers as

$$\begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \mathbf{4.5} & \mathbf{-4.5} \\ \mathbf{-4.5} & \mathbf{4.5} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \mathbf{\frac{1}{2}} \\ \mathbf{\frac{1}{2}} \end{bmatrix}.$$

You have found the 2 by 2 "element stiffness matrix" and the 2 by 1 "element load vector."

(b) On the first and third intervals, similar integrations give

$$\begin{bmatrix} U_1 \end{bmatrix} \begin{bmatrix} \mathbf{3.5} \end{bmatrix} \begin{bmatrix} U_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U_1 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix};$$

$$\begin{bmatrix} U_2 & U_3 \end{bmatrix} \begin{bmatrix} 5.5 & -5.5 \\ -5.5 & 5.5 \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U_2 & U_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Assuming your calculations and mine are correct, what would be the overall finite element equation $KU = F$? (Not to solve)

*Solution.*

(a)

$$\int_{1/3}^{2/3}(1+x)\left(\frac{U_2-U_1}{1/3}\right)^2 dx = \frac{(1+x)^2}{2}\bigg|_{1/3}^{2/3} 9(U_2-U_1)^2$$

$$= \frac{(\frac{5}{3})^2-(\frac{4}{3})^2}{2}\,9(U_2-U_1)^2$$

$$= \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} 4.5 & -4.5 \\ -4.5 & 4.5 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$\int_{1/3}^{2/3}\delta\left(x-\frac{1}{2}\right)u(x)\,dx = U\left(\frac{1}{2}\right) \qquad \left(\text{NOT} \int U(x)\,dx\,!!\right)$$

$$= \frac{1}{2}(U_1+U_2) \qquad \text{(halfway up)}$$

$$= \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

(b)

$$\int_0^{1/3}(1+x)\left(\frac{U_1}{1/3}\right)^2 = \frac{(1+x)^2}{2}\,9\,U_1^2\bigg|_0^{1/3} = 3.5\,U_1^2$$

$$K = \begin{bmatrix} 8 & -4.5 & 0 \\ -4.5 & 10 & -5.5 \\ 0 & -5.5 & 5.5 \end{bmatrix} \qquad F = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$