Quiz II

18.085 (Dr. Christianson)
6/April/09

PRINTED Name: H.C.

- Do all your work on these pages. No calculators or computers may be used. Notes and the text may be used. The point value (out of 100) of each problem is marked in the margin. Note that some problems are worth much more than others and plan your time accordingly. Please confine your answers to the given pages.

1. (*25 points*) This problem concerns least squares for rectangular matrices. The next page is blank for extra work.

a. (*5 pts*) Let A and $\mathbf{b}$ be given by

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 1 \end{pmatrix},$$

and show $A\mathbf{u} = \mathbf{b}$ has no solution $\mathbf{u}$.

b. (*10 pts*)
For the same A and $\mathbf{b}$, find $\hat{\mathbf{u}}$ minimizing

$$\|A\hat{\mathbf{u}} - \mathbf{b}\|.$$

What is the error $\mathbf{e} = A\hat{\mathbf{u}} - \mathbf{b}$?

c. (*10 pts*)
What is the projection $\mathbf{p} = A(A^TA)^{-1}A^T\mathbf{b}$ of the vector $\mathbf{b}$ onto the column space of the matrix A? Show that $\mathbf{e}^T\mathbf{p} = 0$.

a) $A\vec{u} = \vec{b} \Rightarrow u_1 = 2$ and $2u_1 + u_2 = 2 \Rightarrow u_2 = -2$

This contradicts the next line, $u_1 + u_2 = -1$

Therefore, there is no solution.

b) $\hat{u}$ minimizes $\|A\hat{u} - \vec{b}\| \Rightarrow \hat{u}$ solves the normal equation $A^T A \hat{u} = A^T \vec{b}$. That is,

$$\begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \text{ or } \hat{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The error $\vec{e} = A\hat{u} - \vec{b} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

c) The projection $\vec{p} = A(A^T A)^{-1} A^T \vec{b} = A\hat{u}$, so

$\vec{p} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. Indeed, $\vec{e}^T \vec{p} = -1 - 1 + 2 = 0$.

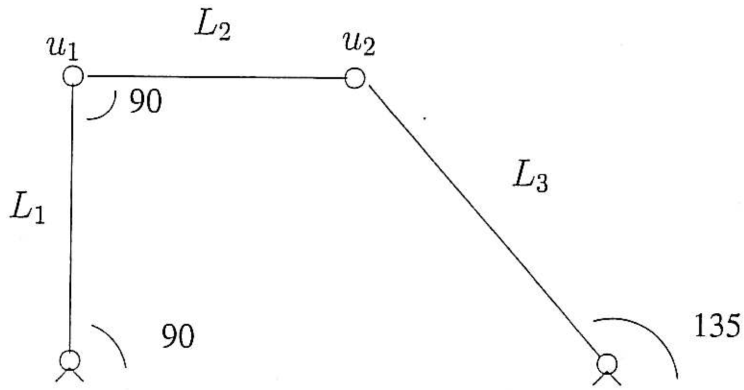


Figure 1: The truss for problem 2.

2. (*30 points*) This problem concerns the truss in Figure 1. Nodes 1 and 2 have the vectors of displacements (u_1^H, u_1^V) and (u_2^H, u_2^V) respectively, with the usual orientation (the other two nodes are fixed). The next page is left blank for extra work.

a. (*10 pts*) Find the elongation matrix A so that

$$\mathrm{e} = A \begin{pmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \end{pmatrix} \tag{1}$$

describes the stretch in bars 1, 2, and 3.

b. (*10 pts*) What are the nullspace solutions to $A\mathrm{u} = 0$? Sketch the motion(s) corresponding to these solutions.

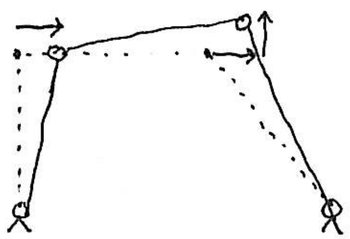
c. (*5 pts*) How many bars do you need to add to eliminate these motions?

d. (*5 pts*) Where would you add it, and what is the new matrix A?

a) $e_1 = u_1^V$
$e_2 = u_2^H - u_1^H$
$e_3 = -\frac{1}{\sqrt{2}} u_2^H + \frac{1}{\sqrt{2}} u_2^V$

, So $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

b) null space is spanned by $\vec{u}_{null} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$



c) Need to add at least one bar.

d)

$A_{new} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

It is easy to see that the columns of A_{new} are independent, so with the new bar, the truss is stable.

3. (*30 points*) This problem concerns the BVP

$$\begin{cases} -u'' = 1, \\ u(0) = u(1) = 0. \end{cases} \tag{2}$$

The next page is blank for extra work.

a. (*10 pts*) Find the exact solution to (2).

b. (*10 pts*) Let $h = 1/3$, and use the piecewise linear hat functions ϕ_1 and ϕ_2 as in your text, and the test functions $V_1 = \phi_1$ and $V_2 = \phi_2$ to calculate the 2×2 finite element matrix K. Calculate the vector $\mathbf{F}$.

c. (*10 pts*) Find the coefficient vector $\mathbf{U}$ satisfying

$$K\mathbf{U} = \mathbf{F}.$$

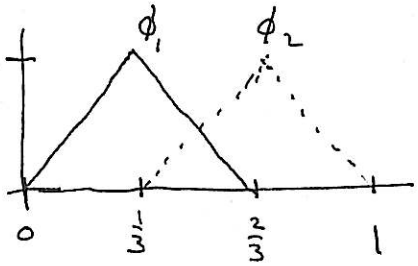
Sketch the graph of $U(x) = U_1\phi_1(x) + U_2\phi_2(x)$ and the graph of the solution $u(x)$ to part (a). What is the maximum error $|U(x) - u(x)|$?

a) $u(x) = -\frac{x^2}{2} + ax + b$. $u(0)=0 \Rightarrow b=0$

$u(1)=0 \Rightarrow a = \frac{1}{2}$

$\Rightarrow u(x) = \frac{1}{2}(x - x^2)$

b)



$$K_{11} = \int_0^1 (\phi_1')^2\,dx = \int_0^{\frac{1}{3}} (3)^2\,dx + \int_{\frac{1}{3}}^{\frac{2}{3}} (-3)^2\,dx$$

$$= \frac{9}{3} + \frac{9}{3} = 6$$

Similarly, $K_{22} = 6$.

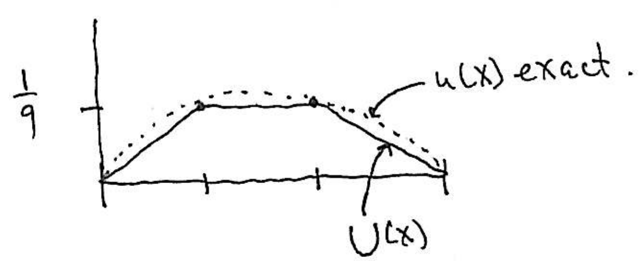
$$K_{12} = \int_0^1 \phi_1'\phi_2' = \int_{1/3}^{2/3} (-3)(3)\,dx = \frac{-9}{3} = -3$$

$K_{21} = K_{12}$, since $V_j = \phi_j$, $j = 1, 2$.

$K = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$. $\quad F_1 = \int_0^1 (\phi_1)(1)\,dx$ = area under $\phi_1 = \frac{1}{3}$

Similarly, $F_2 = \frac{1}{3}$, so $\vec{F} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$.

c) $K\vec{U} = \vec{F} \Rightarrow \vec{U} = \begin{bmatrix} 1/9 \\ 1/9 \end{bmatrix}$. $U(x) = \begin{cases} \frac{1}{3}x, & 0 \le x \le \frac{1}{3} \\ \frac{1}{9}, & \frac{1}{3} \le x \le \frac{2}{3} \\ -\frac{1}{3}x + \frac{1}{3}, & \frac{2}{3} \le x \le 1 \end{cases}$



Maximum of $|u(x) - U(x)|$ is $\frac{1}{72}$, which occurs at $x = \frac{1}{6}, \frac{1}{2}$, and $\frac{5}{6}$.

4. (*15 points*) This problem concerns the perturbed BVP

$$\begin{cases} -u'' + pu = 1, \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

where the parameter $p > 0$. The next page is blank for extra work.

a. (*5 pts*) Find the exact solution to (3). (Hint: a linear combination of the functions e^{ax} and e^{-ax} can also be written as a linear combination of the functions $\sinh ax$ and $\cosh ax$.)

b. (*5 pts*) You calculated the exact solution $u_0(x)$ to the unperturbed problem ($p = 0$) in Problem 3. Using the ansatz $u_{\text{approx}}(x) = u_0(x) + pv(x)$, plug into (3) and discard terms with p^2 to get a BVP for the function v. Solve for v.

c. (*5 pts*) (Do this part only after you have double-checked the rest of your quiz.) Expand the exact solution from part 1 in a Taylor polynomial about $p = 0$ and verify that your approximate solution agrees with the exact solution up to p^2.

a) $-u''+pu=0 \Rightarrow$ characteristic equation $-r^2+p=0 \Rightarrow$

$r=\pm\sqrt{p}$. $p>0 \Rightarrow$ r is real.

$$-u''+pu=1 \rightarrow u=\frac{1}{p}+a_1e^{\sqrt{p}x}+b_1e^{-\sqrt{p}x}$$

$$=\frac{1}{p}+a\cosh\sqrt{p}x+b\sinh\sqrt{p}x$$

for different a,b.

$$0=u(0)=\frac{1}{p}+a \Rightarrow a=-\frac{1}{p}$$

$$0=u(1)=\frac{1}{p}-\frac{1}{p}\cosh\sqrt{p}+b\sinh\sqrt{p}x$$

$$\Rightarrow b=\frac{1}{p}\left(\frac{-1+\cosh\sqrt{p}}{\sinh\sqrt{p}}\right)$$

$$u(x)=\frac{1}{p}-\frac{1}{p}\cosh\sqrt{p}x+\frac{1}{p}\left(\frac{-1+\cosh\sqrt{p}}{\sinh\sqrt{p}}\right)\sinh\sqrt{p}x$$

b) $u_{approx}=\frac{1}{2}(x-x^2)+pv(x)$. Plug in:

$$-u_{approx}''+pu_{approx}=1-pv''+\frac{p}{2}(x-x^2)+\cancel{p^2v}=1$$

Cancelling 1 from each side and looking at terms of order p^1 (discarding p^2) yields the equation

$$\begin{cases} v''=\frac{1}{2}(x-x^2)\\ v(0)=v(1)=0 \end{cases}$$

This has solution $v=\frac{x^3}{12}-\frac{x^4}{24}-\frac{x}{24}$.

Hence $u_{approx}=\frac{1}{2}(x-x^2)+p\left(\frac{x^3}{12}-\frac{x^4}{24}-\frac{x}{24}\right)$

c) using $u(x)$ from part (a),

$$\frac{1}{p} - \frac{1}{p}\cosh\sqrt{p}x + \frac{1}{p}\left(\frac{-1+\cosh\sqrt{p}}{\sinh\sqrt{p}}\right)\sinh\sqrt{p}x$$

$$= \frac{1}{p}\left[1 - \left(1 + \frac{px^2}{2} + \frac{p^2x^4}{24} + O(p^3)\right) + \left(\frac{-1 + \left(1 + \frac{p}{2} + \frac{p^2}{24} + O(p^4)\right)}{p^{1/2} + \frac{p^{3/2}}{6} + \frac{p^{5/2}}{120} + O(p^{7/2})}\right)\right.$$

$$\left.\cdot\left(p^{\frac{1}{2}}x + \frac{p^{3/2}x^3}{6} + \frac{p^{5/2}x^5}{120} + O(p^{7/2})\right)\right]$$

$$= \frac{1}{p}\left[-\frac{px^2}{2} - \frac{p^2x^4}{24} + \left(\frac{p}{2} + \frac{p^2}{24} + O(p^4)\right)\left(1 - \frac{p}{6} + O(p^2)\right)p^{-\frac{1}{2}}\right.$$

$$\left.\cdot\left(p^{\frac{1}{2}}x + \frac{p^{3/2}x^3}{6} + O(p^{5/2})\right) + O(p^3)\right]$$

$$= \frac{1}{p}\left[-\frac{px^2}{2} - \frac{p^2x^4}{24} + \frac{px}{2} + \frac{p^2}{24}x - \frac{p^2}{12}x + \frac{p^2}{12}x^3 + O(p^3)\right]$$

$$= \frac{1}{2}(x-x^2) + p\left(-\frac{x^4}{24} - \frac{x}{24} + \frac{x^3}{12}\right) + O(p^3) \quad \checkmark$$