18.085 Quiz 1 October 2, 2006 Professor Strang

Your PRINTED name is: **SOLUTIONS** **Grading** **1** **2** **3**

1) **(36 pts.)** (a) Suppose $u(x)$ is linear on each side of $x = 0$, with slopes $u'(x) = A$ on the left and $u'(x) = B$ on the right:

$$u(x) = \begin{cases} Ax & \text{for } x \le 0 \\ Bx & \text{for } x \ge 0 \end{cases}$$

What is the second derivative $u''(x)$? Give the answer at every x.

(b) Take discrete values U_n at all the whole numbers $x = n$:

$$U_n = \begin{cases} An & \text{for } n \le 0 \\ Bn & \text{for } n \ge 0 \end{cases}$$

For each n, what is the second difference $\Delta^2 U_n$? Using coefficients $1, -2, 1$ (notice signs!) give the answer $\Delta^2 U_n$ at every n.

(c) Solve the differential equation $-u''(x) = \delta(x)$ from $x = -2$ to $x = 3$ with boundary values $u(-2) = 0$ and $u(3) = 0$.

(d) Approximate problem (c) by a difference equation with $h = \Delta x = 1$. What is the matrix in the equation $K\,U = F$? What is the solution U?

Solutions.

(a) $u''(x) = (B - A)\,\delta(x)$. This is not $B - A$ at $x = 0$.

(b) $$\Delta^2 u_n = \begin{cases} B - A & n = 0 \\ 0 & n \neq 0 \end{cases}$$

(c) $$u(x) = \begin{cases} 3/5\,(x+2) & -2 \le x \le 0 \\ 2/5\,(3-x) & 0 \le x \le 3 \end{cases}$$

(d) $$K_{+++} = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \qquad U = \begin{bmatrix} 3/5 \\ 6/5 \\ 4/5 \\ 2/5 \end{bmatrix} \qquad \begin{matrix} \text{fixed-} \\ \text{fixed} \end{matrix}$$

U lies right on the graph of $u(x)$

2) **(24 pts.)** A symmetric matrix K is "*positive definite*" if $u^{\mathrm{T}}Ku > 0$ for every nonzero vector u.

(a) Suppose K is positive definite and u is a (nonzero) eigenvector, so $Ku = \lambda u$. From the definition above *show that* $\lambda > 0$. What solution $u(t)$ to $\dfrac{du}{dt} = Ku$ comes from knowing this eigenvector and eigenvalue?

(b) Our second-difference matrix K_4 has the form $A^{\mathrm{T}}A$:

$$K_4 = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$$

Convince me how $K_4 = A^{\mathrm{T}}A$ proves that $u^{\mathrm{T}}K_4u = u^{\mathrm{T}}A^{\mathrm{T}}Au > 0$ for every nonzero vector u. (Show me why $u^{\mathrm{T}}A^{\mathrm{T}}Au \geq 0$ and why > 0.)

(c) This matrix is positive definite for which b? Semidefinite for which b? What are its pivots??

$$S = \begin{bmatrix} 2 & b \\ b & 4 \end{bmatrix}$$

Solutions. (8 pts each)

(a) "… *show that* $\lambda > 0$"

$$\begin{aligned} Ku &= \lambda u \\ u^{\mathrm{T}} K u &= \lambda u^{\mathrm{T}} u \end{aligned} \qquad \text{so} \quad \lambda = \frac{u^{\mathrm{T}} K u}{u^{\mathrm{T}} u} > 0$$

"What solution $u(t)$ …"

$$u(t) = e^{\lambda t}\, u$$

(b) $u^{\mathrm{T}} A^{\mathrm{T}} A u = (Au)^{\mathrm{T}}(Au) \geq 0$

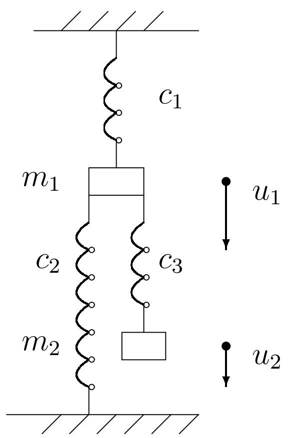
The particular matrix A in this problem has independent columns. The only solution to $Au = 0$ is $u = 0$. So $u^{\mathrm{T}} A^{\mathrm{T}} A u > 0$ except when $u = 0$. Other proofs are *possible*.

(c) Positive definite if $b^2 < 8$

Semidefinite if $b^2 = 8$

Pivots 2 and $4 - \frac{b^2}{2}$

3) **(40 pts.)**



(a) Suppose I measure (with possible error) $u_1 = b_1$ and $u_2 - u_1 = b_2$ and $u_3 - u_2 = b_3$ and finally $u_3 = b_4$. What matrix equation would I solve to find the best least squares estimate $\widehat{u}_1, \widehat{u}_2, \widehat{u}_3$? Tell me the *matrix* and the *right side* in $K\,\widehat{u} = f$.

(b) What 3 by 2 matrix A gives the spring stretching $e = A\,u$ from the displacements u_1, u_2 of the masses?

(c) Find the stiffness matrix $K = A^{\mathrm{T}}CA$. Assuming positive c_1, c_2, c_3 show that K is invertible and positive definite.

Solutions.

(a) (12 points)

$$Au = b \qquad \text{is} \qquad \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\underbrace{A^{\mathrm{T}}A}_{\substack{K \\ \mathit{3pts}}}\,\widehat{u} = \underbrace{A^{\mathrm{T}}b}_{\substack{f \\ \mathit{3pts}}} \qquad \text{is} \qquad \underbrace{\begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix}}_{\substack{K \\ \mathit{3pts\ for} \\ \mathit{numbers}}} \begin{bmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} b_2 - b_1 \\ b_3 - b_2 \\ b_4 + b_3 \end{bmatrix}}_{\substack{f \\ \mathit{3pts}}}$$

(b) (10 points) $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}$

(c) (16 points) $A^{\mathrm{T}}CA = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 & -c_3 \\ -c_3 & c_3 \end{bmatrix}$

Any of these proofs is OK:

(1) $\det = c_1c_3 + c_2c_3 > 0$

(2) $A^{\mathrm{T}}CA$ always positive definite with independent columns in A

(3) other ideas …