SOLUTIONS 18.085 Quiz 1 Fall 2005

1) (a) The incidence matrix A is 12 by 9. Its 4th row comes from edge 4:

$$\text{Row 4 of } A = [\ 0\ \ 0\ \ 0\ \ 0\ \ -1\ \ 1\ \ 0\ \ 0\ \ 0\ ] \qquad \text{(node 5 to node 6)}$$

(b) The 5th column of A indicates edges 3, 4, 9, 10 in and out of node 5:

$$\text{Column 5 of } A = [\ 0\ \ 0\ \ 1\ \ -1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ -1\ \ 0\ \ 0\ ]'$$

The $(5,5)$ entry in $A^{\mathrm{T}}A$ is (column 5)$^{\mathrm{T}}$(column 5) $= 4$. The 5th row of $A^{\mathrm{T}}A$ indicates nodes 2, 4, 6, 8 that are connected to node 5:

$$\text{Row 5 of } A^{\mathrm{T}}A \text{ (also column 5)} = [\ 0\ \ -1\ \ 0\ \ -1\ \ 4\ \ -1\ \ 0\ \ -1\ \ 0\ ].$$

(c) There are 4 independent solutions to $A^{\mathrm{T}}w = 0$ $(n-r = 12-8 = 4 = \text{number of loops})$. The lower left loop uses edges 1, 9, back on 3, back on 7:

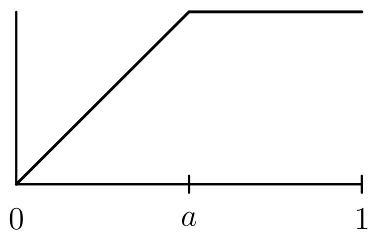
$$w_{\text{loop}} = [\ 1\ \ 0\ \ -1\ \ 0\ \ 0\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ ]'$$

(d) $A^{\mathrm{T}}A$ is **not** positive definite because $Au = 0$ for $u = [\ 1\ \ 1\ \ \ldots\ \ 1\ ]' = \texttt{ones}(9,1)$.

2) (a) The equation $-u'' = \delta(x-a)$ with $u(0)=0$ and $u'(1)=0$ is solved by

$$u(x) = \begin{cases} x & \text{for } x \le a \\ a & \text{for } x \ge a \end{cases}$$

The slope drops from 1 to 0. The graph shows linear displacement above the load, constant below.



(b) As $a \to 1$ the displacement becomes $u(x) = x$. (Notice that this limit doesn't satisfy $u'(1) = 0$.) As $a \to 0$ the displacement becomes $u(x) = 0$ everywhere (the bar hangs free).

(c) The matrix equation (notice the first and last row) will look like

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ u_N \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 1 \end{bmatrix}$$

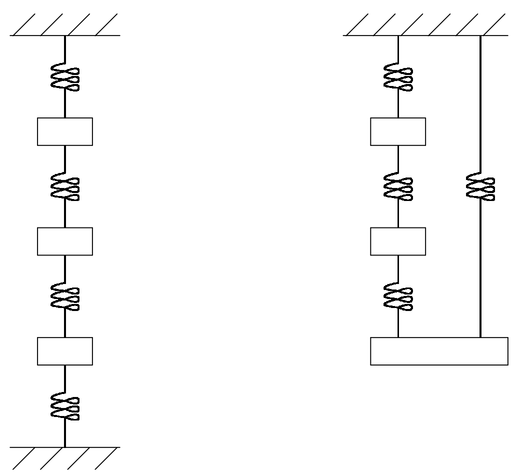
I put the load at the bottom. I should have divided the left side by h^2 and the right side by h! The solution is the last column of the inverse matrix. That column increases linearly just like the continuous case $u(x) = x$:

$$\text{discrete } u = h \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ N \end{bmatrix} \quad \text{and perfection if } Nh = 1.$$

3) (a) If all measurements are correct, then after three steps we reach $u_3 = b_1 + b_2 + b_3 = b_4$. [If you add the first three equations you get $u_3 = b_1 + b_2 + b_3$ and this must equal b_4 for an exact solution.] The equation $A^{\mathrm{T}} A \widehat{u} = A^{\mathrm{T}} b$ for the best estimates has

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{\mathrm{T}} A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{\mathrm{T}} b = \begin{bmatrix} b_1 - b_2 \\ b_2 - b_3 \\ b_3 + b_4 \end{bmatrix}$$

(b) The picture has 3 masses and 4 springs with constants $c_1 = c_2 = c_3 = c_4 = 1$. The forces on the masses should be $f = A^{\mathrm{T}} b$ (not just b). The left figure correctly matches the four original equations. The right figure gives the same $A^{\mathrm{T}} A$ (full credit for that figure also, since professors must get 100% by definition).



(c) The statistically best estimate takes the matrix $C = \texttt{diag}(1/\sigma_1^2, 1/\sigma_2^2, 1/\sigma_3^2, 1/\sigma_4^2) = (c_1, c_2, c_3, c_4)$. Then $A^{\mathrm{T}} C A \widehat{u} = A^{\mathrm{T}} C b$ is

$$\begin{aligned} (c_1 + c_2)\widehat{u}_1 - c_2\widehat{u}_2 &= c_1 b_1 - c_2 b_2 \\ -c_2\widehat{u}_1 + (c_2 + c_3)\widehat{u}_2 - c_3\widehat{u}_3 &= c_2 b_2 - c_3 b_3 \\ -c_3\widehat{u}_2 + (c_3 + c_4)\widehat{u}_3 &= c_3 b_3 + c_4 b_4 \end{aligned}$$

If $\sigma_4 \to \infty$ and $c_4 \to 0$, the first 3 equations are solved exactly to give $u_1 = b_1$ and $u_2 = b_1 + b_2$ and $u_3 = b_1 + b_2 + b_3$.