18.085 Quiz 1 October 4, 2004 Professor Strang

Your name is: **SOLUTIONS**

Grading 1.
2.
3.
4.

OPEN BOOK EXAM

Write solutions onto these pages!

Circles around short answers please!!

1) **(32 pts.)** This problem is about the symmetric matrix

$$H = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(a) By elimination find the triangular L and diagonal D in $H = LDL^{\mathrm{T}}$.

(b) What is the smallest number q that could replace the corner entry $H_{33} = 1$ and still leave H positive *semi*definite? $q =$ _____

(c) H comes from the 3-step framework for a hanging line of springs:

displacements $\xrightarrow{A}$ elongations $\xrightarrow{C}$ spring forces $\xrightarrow{A^{\mathrm{T}}}$ external force f

What are the specific matrices A and C in $H = A^{\mathrm{T}}CA$?

(d) What are the requirements on any m by n matrix A and any symmetric matrix C for $A^{\mathrm{T}}CA$ to be positive definite?

1. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 1/3 \end{bmatrix}$

(a) $H = LDL^{\mathrm{T}}$ with $L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 1/3 \end{bmatrix}$

(b) If $q = \frac{2}{3} = H_{33}$ then H is only semidefinite.

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and $C = I$

(d) A must have n *independent columns* (full rank n)

C must be *positive definite* (to make $A^{\mathrm{T}}CA$ pos def for all those A)

2) **(24 pts.)** Suppose we make three measurements b_1, b_2, b_3 at times t_1, t_2, t_3. They would fit exactly on a straight line $b = C + Dt$ if we could solve

$$\begin{aligned} C &+ Dt_1 = b_1 \\ C &+ Dt_2 = b_2 \\ C &+ Dt_3 = b_3\,. \end{aligned}$$

(a) If b_1, b_2, b_3 are equally reliable **what are the equations** for the best values $\widehat{C}$ and $\widehat{D}$? *Don't solve the equations—write them in terms of t's and b's* (not just some letter A).

(b) Suppose the errors in b_1, b_2, b_3 are independent with variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ (covariances = 0 because independent). **Find the new equations for the best $\widehat{C}$ and $\widehat{D}$. Use** t's, b's **and** $c_i = 1/\sigma_i^2$—by method ① or ②

① Remember how the covariance matrix Σ (called V in the book) enters the equations

② Divide the three equations above by $\sigma_1, \sigma_2, \sigma_3$. Then do ordinary least squares because the rescaled errors have variances = 1.

(c) Suppose $\sigma_1 = 1, \sigma_2 = 1$, but $\sigma_3 \to \infty$ so the third measurement is (*exactly reliable*) (*totally unreliable*) CROSS OUT ONE.

In this case the best straight line goes through which points?

2.

(a) $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$ is $\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\begin{aligned} 3\,\widehat{C} \quad &+ \quad (t_1+t_2+t_3)\,\widehat{D} \quad = \quad b_1+b_2+b_3 \\ (t_1+t_2+t_3)\,\widehat{C} \quad &+ \quad (t_1^2+t_2^2+t_3^2)\,\widehat{D} \quad = \quad t_1b_1+t_2b_2+t_3b_3 \end{aligned}$$

(b) $A^{\mathrm{T}}\Sigma^{-1}A\widehat{x} = A^{\mathrm{T}}\Sigma^{-1}b$ OR $A^{\mathrm{T}}CA\widehat{x} = A^{\mathrm{T}}Cb$ with $c_i = 1/\sigma_i^2$

$$\begin{aligned} (c_1+c_2+c_3)\,\widehat{C} \quad &+ \quad (c_1t_1+c_2t_2+c_3t_3)\,\widehat{D} \quad = \quad c_1b_1+c_2b_2+c_3b_3 \\ (c_1t_1+c_2t_2+c_3t_3)\,\widehat{C} \quad &+ \quad (c_1t_1^2+c_2t_2^2+c_3t_3^2)\,\widehat{D} \quad = \quad c_1t_1b_1+c_2t_2b_2+c_3t_3b_3 \end{aligned}$$

(c) $\sigma_3 \to \infty$ TOTALLY UNRELIABLE

The best line goes through the points (t_1, b_1) and (t_2, b_2)

3) **(20 pts.)** (a) Suppose $\frac{du}{dx}$ is approximated by a *centered* difference:

$$\frac{du}{dx} \approx \frac{u(x+\Delta x) - u(x-\Delta x)}{2\,\Delta x}$$

With equally spaced points $x = h, 2h, 3h, 4h, 5h = \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$ and zero boundary conditions $u_0 = u_6 = 0$, *write down* the 3 by 3 centered first difference matrix Δ:

$$(\Delta u)_i = \frac{u_{i+1} - u_{i-1}}{2h}\,.$$

Show that this matrix Δ is *singular* (because 3 is odd) by solving $\Delta u = 0$.

(b) Removing the last row and column of a positive definite matrix K always leaves a positive definite matrix L. Why? Explain using one of the tests for positive definiteness.

3.

(a) $\Delta = \dfrac{1}{2h}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \Delta u = 0 \text{ for } u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b)

1. There is one *less* pivot and determinant to test.
2. $x^{\mathrm{T}} L x = \begin{bmatrix} x^{\mathrm{T}} & 0 \end{bmatrix} K \begin{bmatrix} x \\ 0 \end{bmatrix} > 0$ for $x \neq 0$
3. $\lambda_{\mathbf{min}}(L) \geq \lambda_{\mathbf{min}}(K) > 0$. This important fact comes from $\lambda_{\mathbf{min}}(K) = \min \frac{x^{\mathrm{T}} K x}{x^{\mathrm{T}} x}$ **the Rayleigh quotient**.

Physically, if you hold the last mass in place then the natural frequencies of a line of springs will (increase) (decrease), as violinists know.

4) **(24 pts.)** The equation to solve is $-u'' + u = \delta(x - \frac{1}{2})$ with a unit point load at $x = \frac{1}{2}$ and zero boundary conditions $u(0) = u(1) = 0$.

(a) Solve $-u'' - u = 0$ starting from $x = 0$ with $u(0) = 0$. There will be one arbitrary constant A. Replace x by $1 - x$ in your answer, to solve $-u'' - u = 0$ ending at $u(1) = 0$ with arbitrary constant B.

(b) Use the "jump conditions" at $x = \frac{1}{2}$ to find A and B.

4.

(a) $u(x) = \begin{cases} A \sin x & \text{for } x \le \frac{1}{2} \\ B \sin(1-x) & \text{for } x \ge \frac{1}{2} \end{cases}$

(b) At $x = \frac{1}{2}$: $\quad A \sin \frac{1}{2} = \quad B \sin \frac{1}{2} \quad$ so $A = B$

Drop in slope : $\quad A \cos \frac{1}{2} = -B \cos \frac{1}{2} + 1 \quad$ so $A = (2 \cos \frac{1}{2})^{-1}$