Quiz I

18.085 (Dr. Christianson)
2/March/09 **PRINTED Name:** ______________________________

- Do all your work on these pages. No calculators or computers may be used. Notes and the text may be used. The point value (out of 100) of each problem is marked in the margin.

1. (*25 points*) This problem concerns solving equations with point masses (delta functions). The next page is blank for extra work.

a. (*10 pts*) Solve the following equation for $u(x)$ with a point mass at $x = a$:

$$\begin{cases} -\frac{d^2}{dx^2} u = \delta(x-a) \\ u(0) = 1, \ u'(1) = -1. \end{cases}$$

b. (*5 pts*)
Use part (a) to solve the following equation for $U(x)$

$$\begin{cases} -\frac{d^2}{dx^2} U = x^2 \\ U(0) = 1, \ U'(1) = -1. \end{cases}$$

(Hint: If $G(x,a)$ is the solution to part (a), then

$$U(x) = \int_0^1 G(x,a) a^2 \, da + \text{null solution}.$$

Don't forget to check the boundary conditions and add an appropriate nullspace solution.)

c. (*10 pts*) Solve the following matrix equation for $a = 1$ and also for $a = 2$:

$$K_3 \mathbf{u} = \delta_a,$$

where as usual

$$K_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

and

$$\delta_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

2. (*25 points*) Consider the function $f(x, y) = 4x^2 - 4xy + 2cy^2$.

a. (*15 pts*) Where does f have critical points (the points where $\nabla f = 0$)? For what values of c does f have a minimum? a maximum? a trough (positive or negative semi-definite)? a saddle point?

b. (*10 pts*) For the value of c giving a trough (the semi-definite case), along which line does the bottom of the trough go (the line where $f(x, y) = 0$)?

3. (*25 points*) This problem concerns SVD and singular values. The next page is blank for extra work.

a. (*10 pts*) Compute the factors U, Σ, V in the SVD for the 1×2 matrix $A = [1 \quad -2]$.

b. (*10 pts*) For the matrix M given by

$$M = \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & \frac{\sqrt{5}}{\sqrt{3}} \end{pmatrix},$$

compute the condition number $c(M) = \sigma_{\max}/\sigma_{\min}$.

c. (*5 pts*) Find vectors $\mathbf{u}$, $\mathbf{f}$, $\Delta\mathbf{u}$, and $\Delta\mathbf{f}$ for which the condition number gives a sharp bound:

$$\frac{\|\Delta\mathbf{u}\|}{\|\mathbf{u}\|} = c(M)\frac{\|\Delta\mathbf{f}\|}{\|\mathbf{f}\|}.$$

(Hint: think in terms of eigenvectors of $M^{\mathrm{T}}M$ or MM^{T}.)

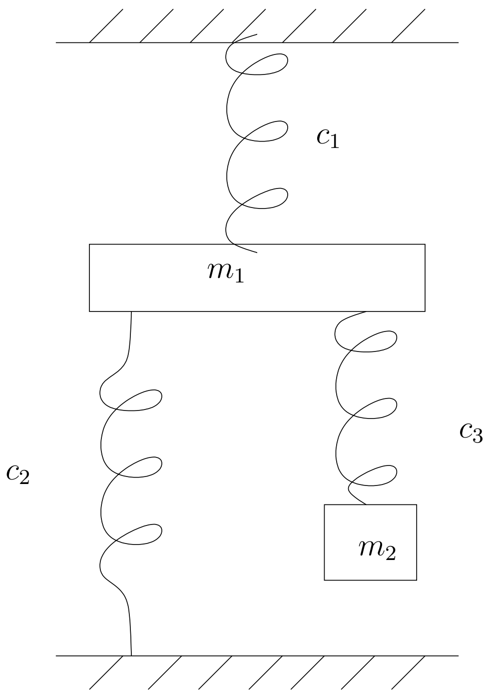

Figure 1: The spring/mass setup for Problem 4.

4. (*25 points*) This problem concerns the spring/mass setup in Figure 1. The next page is blank for extra work.

a. (*15 pts*) The masses are m_1 and m_2, and the springs have spring constants c_1, c_2 and c_3 respectively. There are no external forces (e.g. no gravity). Find the stiffness matrix $K = A^{\mathrm{T}}CA$ in terms of the c_j so that the displacements $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ of the masses satisfies an equation of the form

$$M\mathbf{u}'' + K\mathbf{u} = 0,$$

where M is the matrix of masses as usual.

b. (*10 pts*) For the particular case when $m_1 = 1$, $m_2 = 1/4$, $c_1 = 2$, and $c_2 = c_3 = 1$, find the eigenvalues and eigenvectors of $M^{-1}K$, and find the general solution to Newton's equation (e.g. a combination of pure eigenvector solutions, the normal modes):

$$M\mathbf{u}'' + K\mathbf{u} = 0.$$