1) a)

$$u(x) = Ax + B - R(x-a)$$
$$u(0) = B - R(-a) = B = 1$$
$$u(x) = Ax + 1 - R(x-a)$$
$$u'(x) = A - S(x-a)$$
$$u'(1) = A - 1 = -1 \Rightarrow A = 0$$
$$u(x) = 1 - R(x-a) = \begin{cases} 1 & x < a \\ 1-(x-a), & x > a. \end{cases}$$

b)

$$U(x) = \int_0^1 (1 - R(x-a))a^2\,da + Cx + D$$
$$U(0) = \int_0^1 a^2\,da + D = \tfrac{1}{3} + D = 1 \Rightarrow D = \tfrac{2}{3}$$
$$U'(x) = \int_0^1 (-S(x-a))a^2\,da + C$$
$$U'(1) = -\int_0^1 a^2\,da + C = -\tfrac{1}{3} + C = -1 \Rightarrow C = -\tfrac{2}{3}$$
$$U(x) = \int_0^1 (1 - R(x-a))a^2\,da - \tfrac{2}{3}x + \tfrac{2}{3}$$

c) $j=1$: $\begin{bmatrix} 3/4 \\ 2/4 \\ 1/4 \end{bmatrix}$ $\quad \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3/4 \\ 2/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ✓

$j=2$: $\begin{bmatrix} 2/4 \\ 4/4 \\ 2/4 \end{bmatrix}$ $\quad K_3 \begin{bmatrix} 2/4 \\ 4/4 \\ 2/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ✓.

2. (*25 points*) Consider the function $f(x, y) = 4x^2 - 4xy + 2cy^2$.

a. (*15 pts*) Where does f have critical points (the points where $\nabla f = 0$)? For what values of c does f have a minimum? a maximum? a trough (positive or negative semi-definite)? a saddle point?

$$\nabla f = \begin{bmatrix} 8x - 4y \\ -4x + 4cy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x - y = 0$$
$$-x + cy = 0$$
$$-y + 2cy = 0$$
$$(2c-1)y = 0$$
$c \neq \frac{1}{2} \Rightarrow$ only $(0,0)$

$c = \frac{1}{2} \Rightarrow$ line $y = 2x$.

$$D^2 f = \begin{bmatrix} 8 & -4 \\ -4 & 4c \end{bmatrix}$$

tr $= 8 + 4c$

det $= 32c - 16$ > 0 if $c > \frac{1}{2}$

< 0 if $c < \frac{1}{2}$

$= 0$ if $c = \frac{1}{2}$

$c > \frac{1}{2} \Rightarrow$ tr $> 10 \Rightarrow$ min

$c < \frac{1}{2} \Rightarrow$ saddle

$c = \frac{1}{2} \Rightarrow$ tr $= 10 \Rightarrow$ trough

no maxima.

b. (*10 pts*) For the value of c giving a trough (the semi-definite case), along which line does the bottom of the trough go (the line where $f(x, y) = 0$)?

$c = \frac{1}{2} \Rightarrow$ line $y = 2x$

can also get this from eigenvectors

3) a) $A = [+1 \quad -2]$

$$A^T A = \begin{bmatrix} 1 \\ -2 \end{bmatrix} [1 \quad -2] = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \rightarrow (1-\lambda)(4-\lambda) - 4 = \lambda^2 - 5\lambda = 0$$

$\lambda = 0, 5.$

$\underset{1\times1}{U}$, $\underset{1\times2}{\Sigma}$, $\underset{2\times2}{V}$

5 is only nonzero value

$\Rightarrow \sigma_1 = \sqrt{5}$ and

$\Sigma = [\sqrt{5} \quad 0]$

columns of V are e.vectors of $A^T A$.

$$5 \rightarrow \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \; ; \; 0 \rightarrow \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \rightarrow v_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

only one nonzero σ $\rightarrow$ $u_1 = \dfrac{A v_1}{\sigma_1} = \dfrac{(5/\sqrt{5})}{\sqrt{5}} = 1$

$$A = [1] \; [\sqrt{5} \quad 0] \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} = U \Sigma V^T$$

b) $$M^T M = \begin{bmatrix} \sqrt{3} & 0 \\ 2/\sqrt{3} & \frac{\sqrt{5}}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & \sqrt{5}/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$(3-\lambda)^2 - 4 = 0 \Rightarrow \lambda = \pm 2 + 3 = 5, +1$

$\Rightarrow c(M) = \dfrac{\sqrt{5}}{\sqrt{1}} = \sqrt{5}$

c) $5 \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; 1 \mapsto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$M \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} + 2/\sqrt{3} \\ \sqrt{5}/\sqrt{3} \end{bmatrix} = \underset{\text{length} = \sqrt{10}}{\begin{bmatrix} 5/\sqrt{3} \\ \sqrt{5}/\sqrt{3} \end{bmatrix}} \quad , \; M \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underset{\text{length} = \sqrt{2}}{\begin{bmatrix} 1/\sqrt{3} \\ \sqrt{5}/\sqrt{3} \end{bmatrix}}$$

$\Delta u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Delta f = \begin{bmatrix} 1/\sqrt{3} \\ \sqrt{5}/\sqrt{3} \end{bmatrix}$, $f = \begin{bmatrix} 5/\sqrt{3} \\ \sqrt{5}/\sqrt{3} \end{bmatrix}$.

4) a)

$$e_1 = u_1 \qquad e_2 = -u_1 \qquad e_3 = u_2 - u_1$$

$$\vec{e} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A\vec{u}$$

$$\vec{w} = C\vec{e}, \quad C = \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix}$$

$$K = A^T C A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ -c_2 & 0 \\ -c_3 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + c_2 + c_3 & -c_3 \\ -c_3 & c_3 \end{bmatrix}$$

b) $M = \begin{bmatrix} 1 & \\ & 1/4 \end{bmatrix}$, $K = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$

$$A = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 4 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} = 2K_2$$

eigenvalues: $2 \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $6 \mapsto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$M^{-1/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $M^{-1/2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

are e. vectors for $M^{-1}K = \begin{bmatrix} 4 & -1 \\ -4 & 4 \end{bmatrix}$

soln: $\vec{u} = (A_1 \cos\sqrt{2}t + B_1 \sin\sqrt{2}t) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\quad + (A_2 \cos\sqrt{6}t + B_2 \sin\sqrt{6}t) \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.