18.085 Quiz 1 October 4, 2004 Professor Strang

Your name is: ______________________

Grading 1.
2.
3.
4.

OPEN BOOK EXAM

Write solutions onto these pages!

Circles around short answers please!!

1) **(32 pts.)** This problem is about the symmetric matrix

$$H = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(a) By elimination find the triangular L and diagonal D in $H = LDL^{\mathrm{T}}$.

(b) What is the smallest number q that could replace the corner entry $H_{33} = 1$ and still leave H positive *semi*definite? $q =$ ____

(c) H comes from the 3-step framework for a hanging line of springs:

displacements $\xrightarrow{A}$ elongations $\xrightarrow{C}$ spring forces $\xrightarrow{A^{\mathrm{T}}}$ external force f

What are the specific matrices A and C in $H = A^{\mathrm{T}}CA$?

(d) What are the requirements on any m by n matrix A and any symmetric matrix C for $A^{\mathrm{T}}CA$ to be positive definite?

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2) **(24 pts.)** Suppose we make three measurements b_1, b_2, b_3 at times t_1, t_2, t_3. They would fit exactly on a straight line $b = C + Dt$ if we could solve

$$\begin{aligned} C &+ Dt_1 &= b_1 \\ C &+ Dt_2 &= b_2 \\ C &+ Dt_3 &= b_3\,. \end{aligned}$$

(a) If b_1, b_2, b_3 are equally reliable **what are the equations** for the best values $\widehat{C}$ and $\widehat{D}$? *Don't solve the equations—write them in terms of t's and b's* (not just some letter A).

(b) Suppose the errors in b_1, b_2, b_3 are independent with variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ (covariances $= 0$ because independent). **Find the new equations for the best** $\widehat{C}$ **and** $\widehat{D}$. **Use** t's, b's **and** $c_i = 1/\sigma_i^2$—by method ① or ②

① Remember how the covariance matrix Σ (called V in the book) enters the equations

② Divide the three equations above by $\sigma_1, \sigma_2, \sigma_3$. Then do ordinary least squares because the rescaled errors have variances $= 1$.

(c) Suppose $\sigma_1 = 1, \sigma_2 = 1$, but $\sigma_3 \to \infty$ so the third measurement is (*exactly reliable*) (*totally unreliable*) CROSS OUT ONE.

In this case the best straight line goes through which points?

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3) **(20 pts.)** (a) Suppose $\frac{du}{dx}$ is approximated by a *centered* difference:

$$\frac{du}{dx} \approx \frac{u(x+\Delta x) - u(x-\Delta x)}{2\,\Delta x}$$

With equally spaced points $x = h, 2h, 3h, 4h, 5h = \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$ and zero boundary conditions $u_0 = u_6 = 0$, *write down* the 3 by 3 centered first difference matrix Δ:

$$(\Delta u)_i = \frac{u_{i+1} - u_{i-1}}{2h}\,.$$

Show that this matrix Δ is *singular* (because 3 is odd) by solving $\Delta u = 0$.

(b) Removing the last row and column of a positive definite matrix K always leaves a positive definite matrix L. Why? Explain using one of the tests for positive definiteness.

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4) **(24 pts.)** The equation to solve is $-u'' + u = \delta(x - \frac{1}{2})$ with a unit point load at $x = \frac{1}{2}$ and zero boundary conditions $u(0) = u(1) = 0$.

(a) Solve $-u'' - u = 0$ starting from $x = 0$ with $u(0) = 0$. There will be one arbitrary constant A. Replace x by $1 - x$ in your answer, to solve $-u'' - u = 0$ ending at $u(1) = 0$ with arbitrary constant B.

(b) Use the "jump conditions" at $x = \frac{1}{2}$ to find A and B.

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