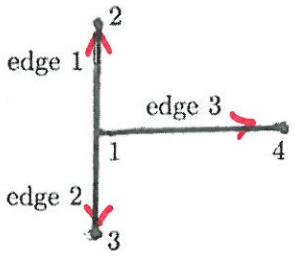
4. In a 4-node graph, the only 3 edges connect node 1 to nodes 2,3,4. So $m = 3$ and $n = 4$.



(a) What is the incidence matrix A? What is its rank?

$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$, rank = 3 (= #nodes - 1)

(b) What is $K = A^T A$ for this graph? Is it positive definite or positive semidefinite and how do you know?

$K = A^T A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

A has dependent columns (since it has rank 3, and 4 columns), so K is not positive definite.

Also, $K \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$, so K cannot be positive definite.

So K is positive semi definite

(explanations why it is pos. semidefinite – not required!: A has maximal possible rank)