18.085 Prof. Strang Quiz 3 December 3, 2003

Your name is: _____ **Grading** 1
2
3
_____
Total

Thank you for taking 18.085, I hope you enjoyed it.

1) **(35 pts.)** Suppose the 2π-periodic $f(x)$ is a half-length square wave:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi/2 \\ -1 & \text{for } -\pi/2 < x < 0 \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$$

(a) Find the Fourier cosine and sine coefficients a_k and b_k of $f(x)$.

(b) Compute $\int_{-\pi}^{\pi} (f(x))^2 \, dx$ as a number and also as an infinite series using the a_k^2 and b_k^2.

(c) DRAW A GRAPH of the integral $I(x) = \int_0^x f(t) \, dt$ from $-\pi$ to π. What are the Fourier coefficients A_k and B_k of $I(x)$?

(d) DRAW A GRAPH of the derivative $D(x) = \frac{df}{dx}$ from $-\pi$ to π. What are the Fourier coefficients of $D(x)$?

(e) If you convolve $D(x) * I(x)$ why do you get the same answer as $f(x) * f(x)$? Not required to find that answer, just explain $D * I = f * f$.

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2) **(33 pts.)** (a) Compute directly the convolution $f * f$ (cyclic convolution with $N = 6$) when $f = (0, 0, 0, 1, 0, 0)$. [You could connect vectors $(f_0, \ldots, f_5)$ with polynomials $f_0 + f_1 w + \cdots + f_5 w^5$ if you want to.]

(b) What is the Discrete Fourier Transform $c = (c_0, c_1, c_2, c_3, c_4, c_5)$ of the vector $f = (0, 0, 0, 1, 0, 0)$? Still $N = 6$.

(c) Compute $f * f$ another way, by using c in "transform space" and then transforming back.

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3) **(32 pts.)** On page 310 the Fourier integral transform of the one-sided decaying pulse $f(x) = e^{-ax}$ (for $x \geq 0$ only) is computed for $-\infty < k < \infty$ as

$$\widehat{f}(k) = \frac{1}{a + ik}\,.$$

(a) Suppose this one-sided pulse is shifted to start at $x = L$:

$$f_L(x) = e^{-a(x-L)} \text{ for } x \geq L, \qquad f_L(x) = 0 \text{ for } x < L\,.$$

Find the Fourier integral transform $\widehat{f}_L(k)$.

(b) Draw a rough graph of the difference $D(x) = F(x) - F_L(x)$ and find its transform $\widehat{D}(k)$. NOW LET $a \to 0$.

What is the limit of $D(x)$ as $a \to 0$?

What is the limit of $\widehat{D}(k)$ as $a \to 0$?

(c) The function $f_L(x)$ is smooth except for a _____ at $x = L$, so the decay rate of $\widehat{f}_L(k)$ is _____ . The convolution $C(x) = f_L(x) * f_L(x)$ has transform $\widehat{C}(k) =$ _____ with decay rate _____ . Then in x-space this convolution $C(x)$ has a _____ at the point $x =$ _____ .

xxx