18.085 Prof. Strang — Quiz 3 Solutions — December 3, 2003

Your name is: ____________________ **Grading** 1
2
3
______
Total

Thank you for taking 18.085, I hope you enjoyed it.

1) **(35 pts.)** Suppose the 2π-periodic $f(x)$ is a half-length square wave:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi/2 \\ -1 & \text{for } -\pi/2 < x < 0 \\ 0 & \text{elsewhere in } [-\pi, \pi] \end{cases}$$

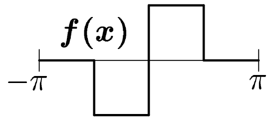
(a) Find the Fourier cosine and sine coefficients a_k and b_k of $f(x)$.

(b) Compute $\int_{-\pi}^{\pi} (f(x))^2\, dx$ as a number and also as an infinite series using the a_k^2 and b_k^2.

(c) DRAW A GRAPH of its integral $I(x)$. (Then $dI/dx = f(x)$ on the interval $[-\pi, \pi]$—choose the integration constant so $I(0) = 0$.) What are the Fourier coefficients A_k and B_k of $I(x)$?

(d) DRAW A GRAPH of the derivative $D(x) = \frac{df}{dx}$ from $-\pi$ to π. What are the Fourier coefficients of $D(x)$?

(e) If you convolve $D(x) * I(x)$ why do you get the same answer as $f(x) * f(x)$? Not required to find that answer, just explain $D * I = f * f$.

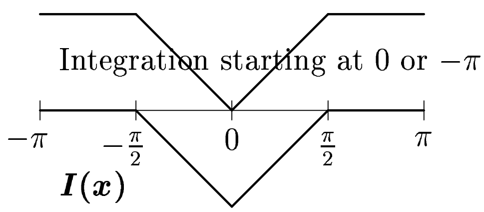
(a) 

$f(x) = $ odd function $= -f(-x)$ so all $a_k = 0$.

Half-interval: $b_k = \frac{2}{\pi} \int_0^{\pi/2} \sin kx\, dx = \frac{2}{\pi} \frac{1 - \cos(k\pi/2)}{k}$.

(b) $\int_{-\pi}^{\pi}(f(x))^2\,dx = \int_{-\pi/2}^{\pi/2} = \pi$. By Parseval this equals $\pi\sum b_k^2$. (Substituting $b_k = \frac{2}{\pi}\left(\frac{1}{1},\frac{2}{2},\frac{1}{3},\frac{0}{4},\ldots\right)$ will give a remarkable formula from $\sum b_k^2 = 1$.)

(c)



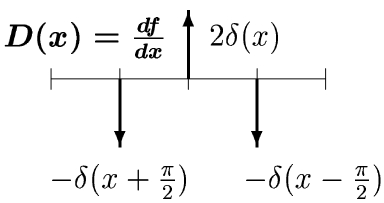
Even function so $B_k = 0$.

Integrating $b_k \sin kx$ gives $-b_k\,\frac{\cos kx}{k}$

so $A_k = \frac{-b_k}{k}$.

The constant term is $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} I(x)\,dx = \frac{3\pi}{8}$ or $-\frac{\pi}{8}$ (integrate starting at 0 or $-\pi$).

(d)



Even function so $B_k = 0$.

Derivative of $b_k \sin kx$ is $kb_k \cos kx$ so $A_k = kb_k$.

Constant term is $A_0 = 0$.

(e) Convolution in x-space is multiplication in k-space. So $f * f$ has complex Fourier coefficients c_k^2 (with factor 2π). And $D(x)*I(x)$ has Fourier coefficients $(ikc_k)(c_k/ik) = c_k^2$ (with same factor). $D * I = f * f$!! Check in x-space:

$$\int_{-\pi}^{\pi} I(t)\,D(x-t)\,dt = \text{ integrate by parts } =$$

$$\int_{-\pi}^{\pi} f(t)\,f(x-t)\,dt + \text{ (boundary term } = 0 \text{ by periodicity)}.$$

The usual minus sign disappears because of 2nd minus sign: $\frac{d}{dt}D(x-t) = -f(x-t)$.

NOTE: I have now learned that we can't just multiply sine coefficients $(kb_k)(-b_k/k)$ because that gives an unwanted minus sign as in $\int \sin t \sin(x-t)\,dt = -\pi\cos x$.

2) **(33 pts.)** (a) Compute directly the convolution $f * f$ (cyclic convolution with $N = 6$) when $f = (0, 0, 0, 1, 0, 0)$. [You could connect vectors $(f_0, \ldots, f_5)$ with polynomials $f_0 + f_1 w + \cdots + f_5 w^5$ if you want to.]

(b) What is the Discrete Fourier Transform $c = (c_0, c_1, c_2, c_3, c_4, c_5)$ of the vector $f = (0, 0, 0, 1, 0, 0)$? Still $N = 6$.

(c) Compute $f * f$ another way, by using c in "transform space" and then transforming back.

With $N = 6$ the complex number $w = e^{2\pi i/6}$ has $w^3 = -1$ and $\overline{w}^3 = -1$ and $w^6 = 1$.

(a) $f = (0, 0, 0, 1, 0, 0)$ corresponds to w^3. Then $f * f$ corresponds to w^6 which is 1. So $f * f = (1, 0, 0, 0, 0, 0)$. (Also seen by circulant matrix multiplication.)

(b) The transform $c = F^{-1} f = \frac{1}{6} \overline{F} f = \frac{1}{6}$ (column of $\overline{F}$ with powers of $\overline{w}^3 = -1$): Then $c = \frac{1}{6}(1, -1, 1, -1, 1, -1)$.

(c) The transform of $f * f$ is $\frac{6}{36}(1^2, (-1)^2, 1^2, (-1)^2, 1^2, (-1)^2) = \frac{1}{6}(1, 1, 1, 1, 1, 1)$.

Multiply that vector v by F to transform back and $Fv = (1, 0, 0, 0, 0, 0)$ as in part (a)!

3) **(32 pts.)** On page 310 Example 3, the Fourier integral transform of the *one-sided* decaying pulse $f(x) = e^{-ax}$ (for $x \geq 0$) $f(x) = 0$ (for $x < 0$) is computed for $-\infty < k < \infty$ as

$$\widehat{f}(k) = \frac{1}{a + ik} \,.$$

(a) Suppose this one-sided pulse is shifted to start at $x = L > 0$:

$$f_L(x) = e^{-a(x-L)} \text{ for } x \geq L, \qquad f_L(x) = 0 \text{ for } x < L \,.$$

Find the Fourier integral transform $\widehat{f}_L(k)$.

(b) Draw a rough graph of the difference $D(x) = f(x) - f_L(x)$, on the whole line $-\infty < x < \infty$. Find its transform $\widehat{D}(k)$. NOW LET $a \to 0$.

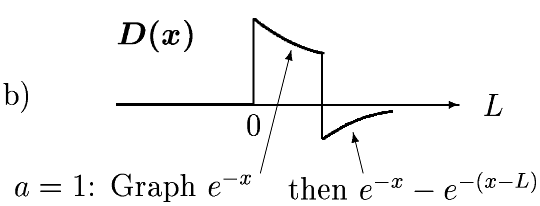
What is the limit of $D(x)$ as $a \to 0$?

What is the limit of $\widehat{D}(k)$ as $a \to 0$?

(c) The function $f_L(x)$ is smooth except for a jump at $x = L$, so the decay rate of $\widehat{f}_L(k)$ is $1/k$. The convolution $C(x) = f_L(x) * f_L(x)$ has transform $\widehat{C}(k) =$ $e^{-i2kL}/(a+ik)^2$ with decay rate $1/k^2$. Then in x-space this convolution $C(x)$ has a corner (= ramp) at the point $x =$ $2L$.

(a) $f_L(x)$ is $f(x - L)$. By the shift rule (page 317) $\widehat{f}_L(k) = e^{-ikL}\widehat{f}(k) = \frac{e^{-ikL}}{a+ik}$.

(b)



As $a \to 0$, $D(x)$ approaches 1 for $0 < x < L$, 0 elsewhere

$\widehat{D}(k) = \frac{1}{a+ik} - \frac{e^{-ikL}}{a+ik}$ approaches $\frac{1-e^{-ikl}}{ik}$ = transform of square pulse.

(c) FILLED IN BLANKS ABOVE