18.085 Prof. Strang Quiz 3 December 7, 2001

Your name is: ______________________________

Grading 1
2
3

Total

Thank you for staying with 18.085. I appreciate it very much. Have a great holiday.

1) **(30 pts.)** Suppose we know a function $f(x)$ between $x = -\pi$ and $x = \pi$. For Fourier series we extend it *periodically* for $|x| > \pi$. For Fourier integrals I will extend it *by zero* for $|x| > \pi$.

(a) How are the Fourier series coefficients c_k related to the Fourier integral transform $\widehat{f}(k)$? Look at the formulas.

(b) How is this integral related to this sum?

$$\int_{-\infty}^{\infty} \widehat{f}(k) e^{ikx} dk \qquad\qquad \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

It seems surprising that the integral uses all k and the sum uses only integer k. (This is the key to Shannon's Sampling Theorem! It works because $f(x)$ is entirely given within the interval $[-\pi, \pi]$)

(c) If $f(x)$ is a hat function (piecewise linear and continuous) how quickly do the coefficients c_k approach zero? What is the decay rate if $f(x) =$ hat function *squared*?

2) **(30 pts.)** Suppose a function $f(x)$ has a known Fourier integral transform $\widehat{f}(k)$. As usual $-\infty < x < \infty$ and $-\infty < k < \infty$. Let $B(x)$ be the box function, equal to 1 for $|x| \le \frac{1}{2}$ (and $B(x) = 0$ for $|x| > \frac{1}{2}$).

(a) If you convolve $f(x)$ with the box function $B(x)$, what is the Fourier integral transform $\widehat{h}(k)$ of $h(x) = f(x) * B(x)$? Do you expect $h(x)$ to be more smooth or less smooth than $f(x)$ (and by how much and why)?

(b) If you consider the derivative $h'(x) = dh/dx$, what is its Fourier integral transform $\widehat{h'}(k)$? Use the standard rule for transform of a derivative.

(c) The convolution of $f(x)$ with the box $B(x)$ is actually given by

$$h(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x-y)dy.$$

Go ahead and take the x-derivative using ordinary calculus (and then integrate $f'(x-y)$ to write $h'(x)$ without any integral). Then take the Fourier transform of this $h'(x)$ by the usual integral formula and hopefully recover the answer in part (b).

3) **(40 pts.)** Suppose you have $N = 8$ values $c_0, c_1, \ldots, c_7$. Let c_{even} and c_{odd} be the "even and odd" subvectors of length 4:

$$c_{\text{even}} = c_0, c_2, c_4, c_6 \qquad c_{\text{odd}} = c_1, c_3, c_5, c_7$$

(a) With $N = 8$ what is the complex number $w_8 = e^{2\pi i/8}$ in its $a+ib$ form? How is w_8 related to the number $w_4 = e^{2\pi i/4}$?

(b) I am looking for a connection between the 8-point DFT of c and the 4-point DFT's of c_{even} and c_{odd}. Look at these particular components:

$$\begin{aligned} Y_1 &= c_0 + c_2 w_4 + c_4 w_4^2 + c_6 w_4^3 \\ Z_1 &= c_1 + c_3 w_4 + c_5 w_4^2 + c_7 w_4^3 \\ y_1 &= c_0 + c_1 w_8 + c_2 w_8^2 + c_3 w_8^3 + c_4 w_8^4 + c_5 w_8^5 + c_6 w_8^6 + c_7 w_8^7 \end{aligned}$$

What combination of Y_1 and Z_1 gives y_1? (This is the key to the Fast Fourier Transform, but just answer directly!).

(c) Suppose $y(n)$ is the average of the four values $x(n), x(n-1), x(n-2), x(n-3)$:

$$y(n) = \frac{1}{4}\sum_{k=0}^{3} x(n-k), \qquad -\infty < n < \infty.$$

This y is a (non-cyclic) convolution of the vector x with what vector h?

(d) In part (c) find the relation between the functions

$$Y(\omega) = \sum y(n)e^{-in\omega} \qquad \text{and} \qquad X(\omega) = \sum x(n)e^{-in\omega}$$

Find an input other than $x = (\ldots, 1, -1, 1, -1, \ldots)$ that leads to the output $y =$ all zeros.