18.085 Prof. Strang Quiz 2 SOLUTIONS November 7, 2003

Your name is: ______________________________ **Grading** 1
2
3

Total

1) **(36 pts.)**

(a) For $-\frac{d^2u}{dx^2} = \delta(x-a)$ with $u(0) = u(1) = 0$, the solution is linear on both sides of $x = a$ (graph = triangle from two straight lines). *What is wrong with the next sentence*? Integrating both sides of $-u'' = \delta(x-a)$, from $x = 0$ to $x = 1$, the area under the triangle graph is 1. The base is 1 so the height must be $u_{\max} = 2$.

(b) If $u(x, a)$ is the **true** solution to that standard problem in part (a), and the load point $x = a$ approaches $x = 1$, *what function does the solution u approach*? Give a physical reason for your answer or a math reason or both.

Integrating $\delta(x-a)$ does give 1. But the area under the graph of $u(x)$ is $\int u(x)\,dx$ and not $\int -u''(x)\,dx$. (The integral of $-u''(x) = \delta(x-a)$ gives the drop in slope $u'(0) - u'(1) = 1$.) So the reasoning is wrong and $u_{\max}$ is not 2. The actual $u_{\max}$ is $(a-1)\,a$, because the true solution has slope $a-1$ up to the load point $x = a$. As that point approaches $a = 1$, *the solution $u(x)$ approaches zero*. Physically, the load is moving close to the support. Then the load causes a smaller and smaller displacement.

2) **(40 pts.)** Which of these 5 equations can be solved?? If the equation has a solution, please find *one*. If not *why not*??

(a)

$$\operatorname{div} \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = 1$$

(b)

$$\operatorname{div} \begin{bmatrix} \partial s/\partial y \\ -\partial s/\partial x \end{bmatrix} = 1$$

(c) $u_{xx}+u_{yy} = 0$ in the unit circle and $u(1,\theta) = \sin 4\theta$ around the boundary

(d) Find a family of curves $u(x,y) = C$ that is everywhere perpendicular to the family of curves $x + x^2 - y^2 = C$.

(e) $d^4u/dx^4 = \delta(x)$ [point load at $x = 0$, not requiring boundary conditions, any solution $u(x)$ is OK].

(a) This is Poisson's equation $u_{xx} + u_{yy} = 1$. One solution is $u(x,y) = \frac{1}{2}x^2$.

(b) This equation is $\frac{\partial^2 s}{\partial x\,\partial y} - \frac{\partial^2 s}{\partial y\,\partial x} = 1$. No solution since $s_{xy} = s_{yx}$.

(c) $u(r,\theta) = r^4 \sin 4\theta$ solves Laplace's equation and reduces to $\sin 4\theta$ on the unit circle $r = 1$.

(d) The function $x+x^2-y^2$ is the real part of $(x+iy)+(x+iy)^2$. So we get perpendicular curves from the imaginary part $y + 2xy = C$.

(e) $u = \begin{cases} 0 & \text{for } x \le 0 \\ x^3/6 & \text{for } x \ge 0 \end{cases}$ has jump of 1 in u''' and $u'''' = \delta(x)$. It comes from integrating $\delta(x)$ *four times.*

3) **(24 pts.)** I want to solve Laplace's equation (really Poisson's equation) in 3D with *no boundaries* (the whole space). The right side is a point load δ at the origin $(0,0,0)$. So $-\text{div}(\text{grad } u) = \delta$.

(a) Integrate both sides over a sphere of radius R around the origin:

$$\iiint -\text{div}(\text{grad } u)\, dxdydz = 1\,.$$

Use the divergence theorem (or Gauss-Green) to transform that triple integral into an integral on the surface of the sphere of radius R.

(b) What is the normal vector n in that integral? Knowing that u must be radially symmetric, $\frac{\partial u}{\partial r}$ is a constant on the sphere of radius R. What is that constant?

(c) So what is $u(r)$?

(a) The divergence theorem transforms to a double integral for the flux through the sphere:

$$\iint (\text{grad } u) \cdot n\, dS = 1\,.$$

(b) Since n is the outward (radial) unit vector, $(\text{grad } u) \cdot n$ is the same as $\partial u/\partial r$. It is constant on the sphere (which has area $4\pi R^2$) so its value is $1/4\pi R^2$.

(c) If $\partial u/\partial r = 1/4\pi r^2$ whenever $r = R$, then $u(r) = 1/4\pi r$.