18.085 Prof. Strang — Quiz 2 — November 5, 2001

Your name is: ______________________________

Grading	1
	2
	3
Total	

1) **(30 pts.)** A second-order equation comes from two simpler equations:

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \text{ comes from } -\frac{dw}{dx} = f(x) \text{ and } c(x)\frac{du}{dx} = w(x).$$

The boundary conditions for a fixed-free rod are $u(0) = 0$ and $w(1) = 0$. Suppose there is a unit point load at $x = \frac{2}{3}$, so $f(x) = \delta(x - \frac{2}{3})$. Suppose $c(x) = 6$ for $x \le \frac{1}{3}$, and at that point we change to a different material with $c(x) = 12$ for $x \ge \frac{1}{3}$.

(a) Solve $-\frac{dw}{dx} = f(x)$ and *graph* the solution $w(x)$.

(b) Now solve $c(x)\frac{du}{dx} = w(x)$ and *graph* the solution $u(x)$.

(c) Which of the functions $u(x)$ and du/dx and $w(x)$ do you expect to be continuous at a point where $c(x)$ jumps (but $f(x)$ is smooth)? Which of those functions do you expect to be continuous at a point where $f(x)$ has a jump (*not* a delta function—and $c(x)$ is smooth)?

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2) **(40 pts.)**

(a) Find the real part u and the imaginary part s as functions of r, θ and also as functions of x, y, for the logarithm of $z^2 = (x+iy)^2$:

$$f(x+iy) = ln[(x+iy)^2] = ln[(re^{i\theta})^2].$$

(b) What is the gradient $v(x,y)$ of this function $u(x,y)$? What are the streamlines of the flow?

(c) Compute the flux $\int v \cdot n \; ds$ through the unit circle in one of these two ways:

- Flux = (stream fcn at end of circle) − (stream fcn at beginning of circle).
- The unit normal n points in the r direction, and arc length is $ds = d\theta$, integrating around the unit circle (polar coordinates).

(d) Compute the flux $\int v \cdot n \; ds$ through a big square of side $2R$ centered at $(0,0)$. You could use one of the ways above (now $ds = dx$ or dy or $-dx$ or $-dy$ around the square). Or you could use the divergence theorem **between the square and the circle**:

- $\int\int$ div $v \; dxdy$ = flux through square minus flux through circle.

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3) **(30 pts.)** We want to approximate $-u_{xx} = 4$ with zero boundary conditions $u(0) = u(1) = 0$ by a finite element equation $KU = F$. Divide the interval from $x = 0$ to $x = 1$ into $N+1$ equal pieces of length $h = (N+1)^{-1}$. On each piece the N trial functions $\phi_j(x)$ and the N test functions $V_k(x)$ are *linear*. Then $\phi_j = V_j = 1$ at the jth meshpoint $x = jh$, and $\phi_j = V_j = 0$ at other meshpoints.

(a) Not yet finite elements: Find the weak form of $-u_{xx} = 4$ including the boundary conditions on $U(x)$ and $V(x)$.

(b) Finite elements: Now substitute $U(x) = \sum_1^N U_j \phi_j(x)$ and $V(x) = \phi_k(x)$ into the weak form. What equation for the numbers U_j comes from computing the integrals in the weak form? This is equation k in the system $KU = F$.

(c) If we add a point load at $x = 1.5h$, half way between two meshpoints, which entries of the matrix K and the vector F will be changed? (Don't compute the changes.)

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