18.085 Exam 1 October 12, 2011 Professor Strang

Your PRINTED name is: ____________________ 1.

Your class number is ____________________ 2.

3.

4.

1. Suppose u_1, u_2, u_3, u_4 are unknowns at meshpoints $x = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$ with $h = \frac{1}{4}$ and boundary value $u_0 = 0$ at one end.

(a) Write down a 4 by 4 backward difference matrix A, which acts on u to produce differences $e_i = (u_i - u_{i-1})/h$.

$$A = \frac{1}{h}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & -4 & 4 \end{pmatrix}$$

(b) Find the symmetric part $S = \frac{1}{2}(A + A^T)$ and the antisymmetric part $AS = \frac{1}{2}(A - A^T)$. What words would you use for this particular S and AS? (AS – on separate page)

$$S = \frac{1}{h}\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

– non-degenerate (invertible)
tridiagonal, const. diagonals, looks like 2-nd difference but with $\frac{1}{h}$ instead of $\frac{2}{h^2}$

(c) Compute $K = A^TCA$ when $C = \text{diag}(c_1, c_2, c_3, c_4)$ is a diagonal matrix with positive c's. Show by some test that K is positive definite. (Explain your test exactly.)

Separate page.

(d) If $f = (0, 0, 0, 1)$ is a unit force pulling the 4th mass (at $x = 1$) downward, what will be the displacement u_4 at the bottom of the line of springs? *What entry of K^{-1} does this tell you?* Use **physical reasoning** with 4 springs, much quicker than solving 4 equations

$f = (0,0,0,1)$ means we have only one mass (m_4). So we can think of this as one long spring with Hooke's constant $c = \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}\right)^{-1}$

Get $u_4 = h^2\left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}\right) = \frac{1}{16} \cdot \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \frac{1}{c_4}\right)$

Since $u = K^{-1}f$, we have $u_4 = (K^{-1})_{4,4}$.

Problem 1

1 b

$$AS = \frac{1}{h}\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

- invertible, tridiagonal, constant diagonals, (second difference matrix) $\cdot$ 2.

1 c

$$K = A^T C A = \frac{1}{h^2}\begin{pmatrix} c_1+c_2 & -c_2 & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & 0 \\ 0 & -c_3 & c_3+c_4 & -c_4 \\ 0 & 0 & -c_4 & c_4 \end{pmatrix}$$

Why K is positive definite:

$K = A^T C A$, A - has independent columns
C has positive entries on the diagonal (and it's a diagonal matrix)

$\Rightarrow$ K is positive definite.

Can also check determinants:

$c_1 + c_2 > 0$

$c_1 c_2 + c_2 c_3 + c_1 c_3 > 0$

$c_1 c_2 c_3 + c_1 c_2 c_4 + c_2 c_3 c_4 + c_1 c_3 c_4 > 0$

$c_1 c_2 c_3 c_4 > 0$