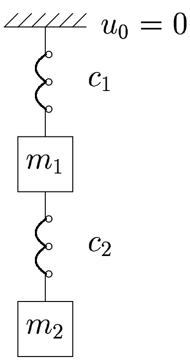
Solutions

18.085 Quiz 1 Professor Strang October 6, 2003

1) **(30 pts.)** A system with 2 springs and masses is **fixed-free**. Constants are c_1, c_2.



(a) Write down the matrices A and $K = A^{\mathrm{T}}CA$.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

(b) Prove by **two tests** (pivots, determinants, independence of columns of A) that this matrix K is (positive definite) (positive semidefinite). **Tell me which two tests you are using**!

Determinants of K: $c_1 + c_2$ (1×1) and $c_1 c_2$ (2×2)

Pivots of K: $c_1 + c_2$ and $c_1 c_2/(c_1 + c_2)$

Independence of columns of A: $(1, -1)$ and $(0, 1)$

All prove positive definiteness of K.

(c) Multiply column times row to compute the "element matrices" K_1, K_2:

$$\begin{aligned} \text{Compute} \quad K_1 &= (\text{column 1 of } A^{\mathrm{T}})(c_1)(\text{row 1 of } A) \\ &= c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \text{Compute} \quad K_2 &= (\text{column 2 of } A^{\mathrm{T}})(c_2)(\text{row 2 of } A) \\ &= c_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

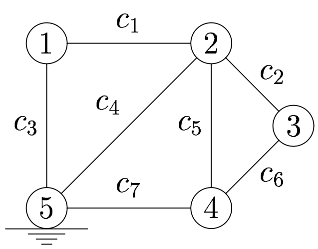
Then $K = K_1 + K_2$. What vectors solve $K_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} C \\ C \end{bmatrix}$$

For those displacements x_1 and x_2, **what is the energy in spring 2**?

Zero (no stretching!)

2) **(33 pts.)** A network of nodes and edges and their conductances $c_i > 0$ is drawn without arrows. Arrows don't affect the answers to this problem; the edge numbers are with the c's. Node 5 is grounded (potential $u_5 = 0$).



(a) List all positions (i, j) of the 4 by 4 matrix $K = A^{\mathrm{T}}CA$ that will have **zero entries**. What is row 1 of K?

No bars node 1 to node 3, node 1 to node 4 (and 3 to 5). So $K_{13} = K_{31} = K_{14} = K_{41} = 0$ (and $K_{\text{unreduced}}$ would have $K_{35} = K_{53} = 0$: not asked).

Row 1 of K comes from bar 1: $[\, c_1 + c_3, -c_1, 0, 0\,]$

(b) Find as many independent solutions as possible to Kirchhoff's Law $A^{\mathrm{T}}y = 0$.

Here we need arrows (sorry) to give consistent signs:

$$y_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \qquad y_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}; \qquad y_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

(c) Is $A^{\mathrm{T}}A$ always positive definite for every matrix A?

No. (A *is any matrix*)

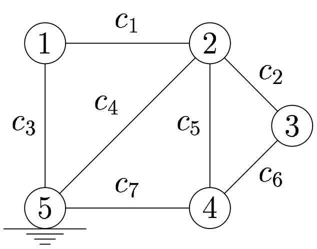
If there is a test on A, what is it?

(A must have independent columns. It can be tall and thin!)

What is the trick that proves $u^{\mathrm{T}}Ku \geq 0$ for every vector u?

$u^{\mathrm{T}}Ku = (u^{\mathrm{T}}A^{\mathrm{T}})C(Au) = e^{\mathrm{T}}Ce = c_1e_1^2 + \cdots + c_me_m^2$.

3) **(37 pts.)** Make the network in Problem 2 into a 7-bar truss! The grounded node 5 is now a supported (but turnable) pin joint, with known displacements $u_5^{\mathrm{H}} = u_5^{\mathrm{V}} = 0$. **All angles are** $45°$ **or** $90°$.



(a) How many rows and columns in the (reduced) matrix A, after we know $u_5^{\mathrm{H}} = u_5^{\mathrm{V}} = 0$?

7 rows (7 bars) and 8 columns (8 unknown u's).

Describe in words (or a picture) all solutions to $Au = 0$.

$Au = 0$ when $u =$ rigid rotation around node 5. (A has a 1-dimensional nullspace.)

If you add 1 bar can A become square and invertible?

Not invertible since rotation is still allowed.

(b) Write out row 2 of A, corresponding to bar 2.

Row 2 $= \begin{bmatrix} 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix}$

Then (row 2) times the column u of displacements has what physical meaning?

(Row 2)u is the infinitesimal stretching of bar 2 in response to the small displacements u.

(c) What is the first equation of $A^{\mathrm{T}}w = f$ (with right side f_1^{H})?

The first equation is the horizontal force balance at node 1. Since y measures stretching (rather than compression) according to our convention, the horizontal force balance at node 1 is $y_1 = -f_1^{\mathrm{H}}$.

Why does $\frac{1}{2}u^{\mathrm{T}}Ku = \frac{1}{2}y^{\mathrm{T}}C^{-1}y$ and what does this quantity represent physically?

$\frac{1}{2}u^{\mathrm{T}}Ku = \frac{1}{2}e^{\mathrm{T}}Ce = \frac{1}{2}y^{\mathrm{T}}C^{-1}y$ represents the internal energy in the 7 bars.