SOLUTIONS TO QUIZ 3 2011

1. (a)
$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{ax}e^{-ikx}\,dx$$
$$= \frac{1}{2\pi}\left.\frac{e^{(a-ik)x}}{a-ik}\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\,\frac{e^{(a-ik)2\pi}-1}{a-ik}.$$

(b) The decay ~~rate~~ is $\frac{1}{k}$, and $f(x)$ has a jump at $x=0$ (and at every multiple of 2π):
$$\frac{df}{dx} = \underset{\text{(NORMAL)}}{a e^{ax}} + \underset{\text{(JUMP)}}{\left(1-e^{2\pi a}\right)\delta(x)}$$

If $a = iN$ (any whole number) then the jump is zero and $f(x) = e^{iNx}$ has only one nonzero Fourier coefficient.

For $a = b + is$,
(c) the energy equation (1.1.31) gives
$$\sum |c_k|^2 = \frac{1}{2\pi}\int |f(x)|^2 dx = \frac{1}{2\pi}\int_0^{2\pi} e^{2bx}\,dx$$
$$= \frac{1}{2\pi}\left.\frac{e^{2bx}}{2b}\right]_0^{2\pi} = \frac{e^{4\pi b}-1}{4\pi b}$$

2. (a) The vector c goes into the first column of the circulant matrix C:

$$C = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} \quad \text{and } Cd = c \circledast d = C\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

(b) The polynomials for $c = (2,0,0,1)$ and $d = (0,0,1,0)$ are

$$c(w) = 2 + w^3 \quad \text{and} \quad d(w) = w^2$$

Then $c(w)d(w) = 2w^2 + w^5 = 2w^2 + w$ matches the answer to (a)

(c) With $N=4$, $$Fd = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ i^2 \\ i^4 \\ i^6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

And $d(w) = w^2$ at $w = 1, i, i^2, i^3$ is exactly $1, -1, 1, -1$.

(d) $$C^2 = \begin{bmatrix} 4 & 4 & 1 & 0 \\ 0 & 4 & 4 & 1 \\ 1 & 0 & 4 & 4 \\ 4 & 1 & 0 & 4 \end{bmatrix}$$ gives convolution with $c \circledast c$ which is the vector $y = (4, 0, 1, 4)$.

(Notice that $(2+w^3)(2+w^3) = 4 + 4w^3 + w^6$ and $w^6 = w^2$)

3. $\hat{f}(k) = \int_0^L e^{-ikx}\,dx - \int_L^{2L} e^{-ikx}\,dx$

$$= \frac{e^{-ikx}}{-ik}\Bigg]_0^L - \frac{e^{-ikx}}{-ik}\Bigg]_L^{2L}$$

$$= \frac{1}{ik}\left(e^{-ik2L} - 2e^{-ikL} + 1\right)$$

also $= \frac{1}{ik}\left(e^{-ikL} - 1\right)^2$

(b) $\hat{f}(k) = 0$ at $k=0$ because $e^{-ikL} = 1 - ikL + O(k^2)$

for small k, $\hat{f}(k) \approx \frac{(-ikL)^2}{-ik}$ approaches zero.

NEW LINE

Put $k=0$ in the general formula (4.5.2) for $\hat{f}(k)$ to get

$$\hat{f}(0) = \int f(x)\,dx = \text{area under the graph}$$

(c) $\frac{df}{dx} = \delta(x) - 2\delta(x-L) + \delta(x-2L)$

The transform of this df/dx is $1 - 2e^{-ikL} + e^{-ik2L}$

(And we divide by ik to check $\hat{f}(k)$ in part (a): It works!)