18.085
Quiz

Four questions.

Answers in the space provided can score points.

NAME:

Question 1

(**25 points**)
Fourier series.

a Consider a Fourier series expansion

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

for

$$f(x) = x^2, \qquad -\pi < x < \pi.$$

Are the coefficients b_k zero? (Answer yes or no, and give a short reason.)

b We find cosine coefficients a_k from a formula of the form $a_k = \text{constant} \times \int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x$. Alternatively, in a complex Fourier series $f(x) = \sum_{-\infty}^{+\infty} c_n e^{inx}$, we find the coefficients from a formula of the form $c_k = \text{constant} \times \int_{-\pi}^{\pi} f(x)e^{-ikx}\mathrm{d}x$. Compared to expanding f in *any* old set of basis functions, circle one phrase that best describes the key property that makes these formulas 'work' (or if you prefer, the key step used to derive them):

harmonic functions Euler's characteristic

antisymmetric basis functions orthogonal basis functions

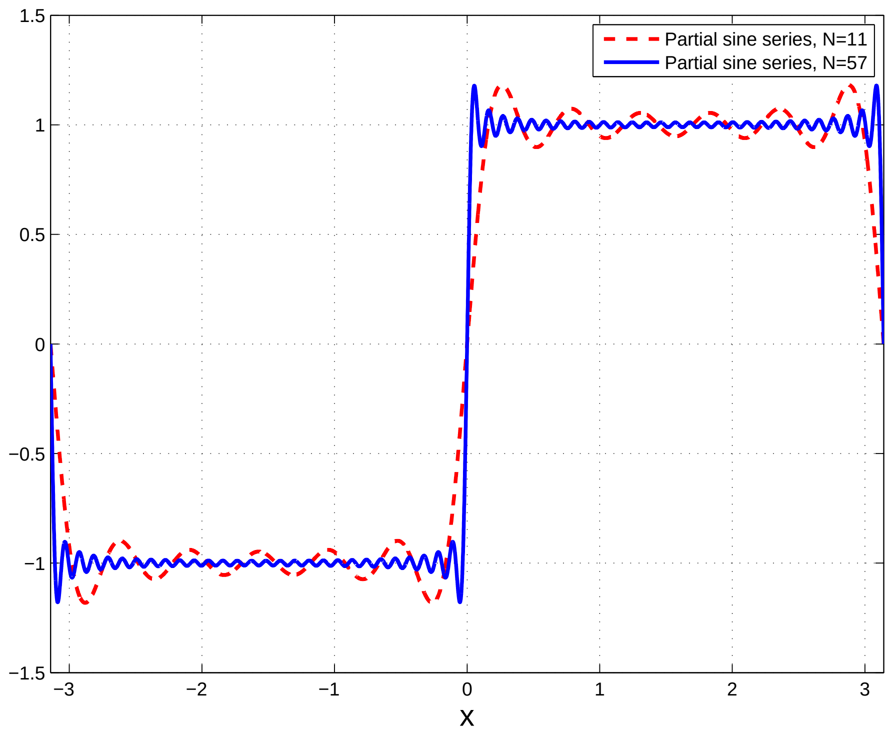
c Circle one option. Compared to the rate of decay of the Fourier coefficients of $f(x)$, we expect the rate of decay of the Fourier coefficients of $\mathrm{d}f/\mathrm{d}x$ to be:

slower, about the same, faster

d Partial sums with N terms from a Fourier sine series expansion for a function $S(x)$ are shown. The rate of decay of the coefficients is slow. Circle one option. Observing the series near $x = 0$, the description of S that seems most likely is:

S is an infinitely differentiable, smooth, analytic function

S has a discontinuity, and we are seeing the Gibbs phenomenon



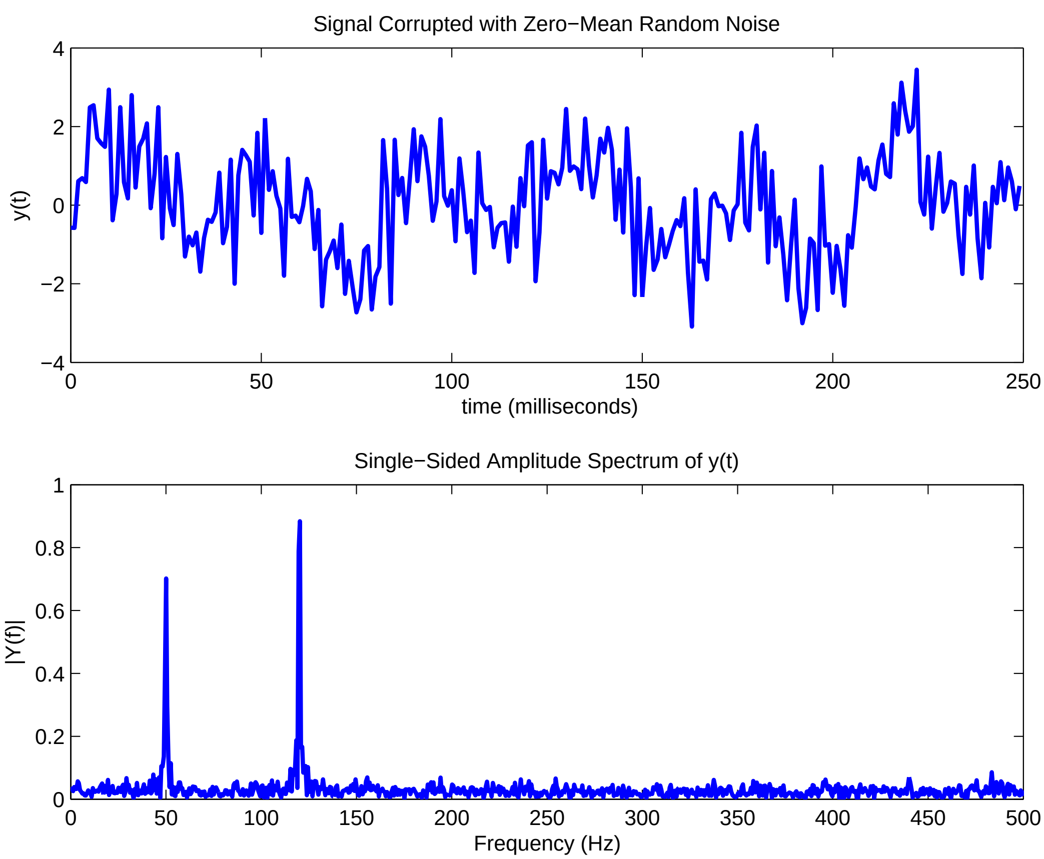
e As shown below, a signal $y(t) = S(t) + \text{noise}$ is detected (top panel), and transformed via FFT in MATLAB (bottom panel). Circle one option. Of the following, the true signal $S(t)$ is most likely to be:

$0.7\sin(50 \times 2\pi t) + \sin(120 \times 2\pi t)$

$\sin(50 \times 2\pi t) + 0.7\sin(120 \times 2\pi t)$

$\sin(2\pi t) + \sin(2\pi t)$

$\sin(10 \times 2\pi t) + 0.7\sin(100 \times 2\pi t)$



Question 2

(25 points)
Stream lines, equipotentials, gradient, divergence

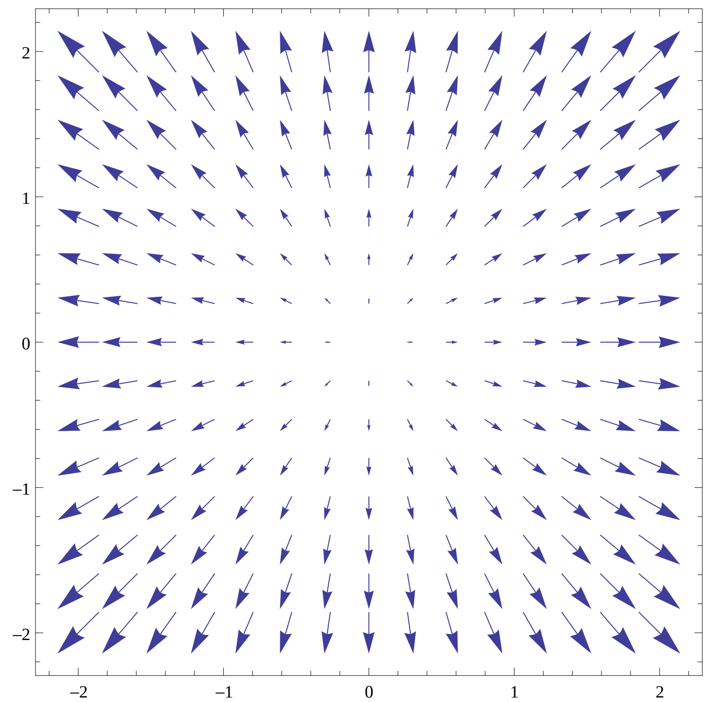
a A vector field is shown below. Circle one phrase that best describes this vector field:

Divergence $= 0$ and
there *is* an associated stream function

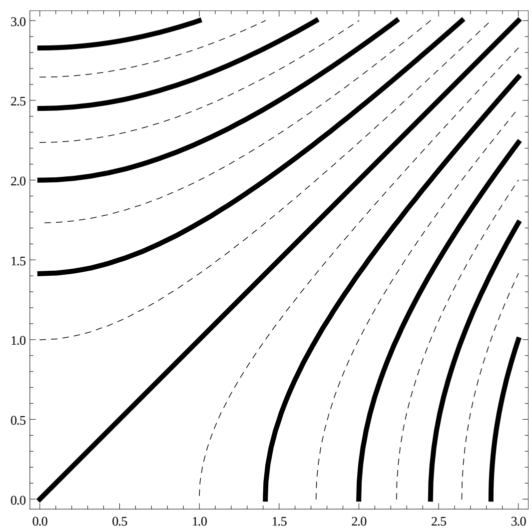
Divergence $\neq 0$ and
there *is* an associated stream function

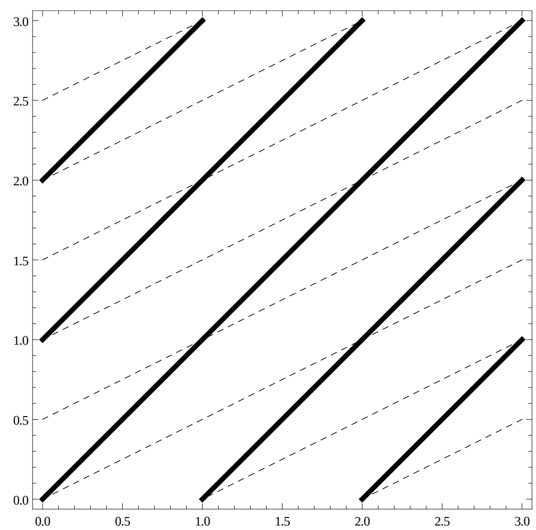
Divergence $= 0$ and
there is *not* an associated stream function

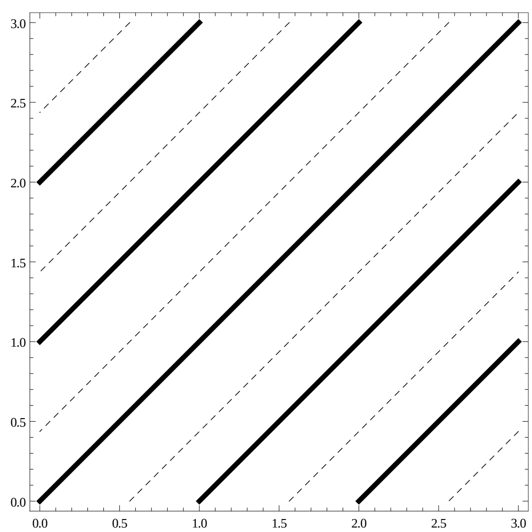
Divergence $\neq 0$ and
there is *not* an associated stream function

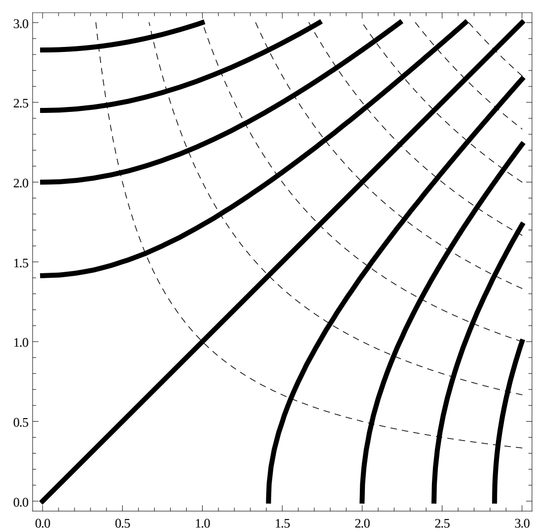


b Consider a nice vector field that has an associated potential function $u(x, y)$ and associated stream function $s(x, y)$, which both satisfy Laplace's equation. Circle one figure that best shows the superposition of the equipotentials ($u =$ constant, dashed) and of the streamlines ($s =$ constant, solid).









c Circle one option that gives the best explanation. The components $(v_1, v_2) = (\partial u/\partial x,\, \partial u/\partial y)$ of a gradient field must pass a zero vorticity test, $\partial v_1/\partial y - \partial v_2/\partial x = 0$, because:

$$\frac{\partial u}{\partial x} = -\frac{\partial s}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{\partial s}{\partial x} \qquad\qquad \frac{\partial^2 u}{\partial x \partial y} = i\frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} \qquad\qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

d Consider the vector field $v = (x + y, x - y)$.

i Is v the gradient of a potential? (Answer yes or no.)

ii If yes, find a potential function. If no, give a reason.

Question 3

(25 points)
Laplace's equation

a Suppose $F(x, y)$ satisfies Laplace's equation: $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$.
True or False:

"$u = \frac{\partial F}{\partial y}$ and $s = \frac{\partial F}{\partial x}$ satisfy the Cauchy-Riemann equations"

b Using polar coordinates, solve Laplace's equation for $u(r, \theta)$ inside the unit circle, with boundary condition

$$u = u_0(\theta) = \sin\theta + \sin 2\theta$$

on the unit circle.

c Suppose $s(x, y)$ is the imaginary part of a nice complex function (if you prefer, nice could mean analytic and okay to think of powers z^n). Find

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2}.$$

d Find the real $u(r, \theta)$ and imaginary $s(r, \theta)$ parts of

$$f(z) = \frac{1}{z} = \frac{1}{x + iy} = \frac{1}{re^{i\theta}}$$

e Find a function $s(r, \theta)$ that satisfies

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} = 0$$

outside the unit circle, and that satisfies the boundary condition $s = s(1, \theta) = y$ on the unit circle. [1]

[1] More than one possible correct answer (some nicer than others) and any one of those receives full credit. Parts [c], [d] and [e] are related.

Question 4

(**25 points**)
Finite elements in 1D with piecewise linear basis functions.

Consider the equation

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 18$$

with fixed-free boundary conditions $u(0) = 0 = u'(1)$.

a **i** What is the weak form?

ii In the weak form, what are the boundary conditions on $u(x)$?

iii In the weak form, what are the boundary conditions on test functions $v(x)$?

b Create a finite element equation $K \;\; U = F$ that models this problem. Take $h = \Delta x = 1/3$. As usual, take basis functions $\phi(x)$ to be hat functions (take two hat functions and one half-hat function), and let test functions be the same as basis functions, i.e. $v(x) = \phi(x)$.

i Find the $(1, 2)$ entry of the stiffness matrix K.

ii The stiffness matrix is the sum of *element matrices*: $K = K_1 + K_2 + K_3$. For example, the element matrix K_2 corresponds to the middle element $[1/3, 2/3]$ of the domain. **Find $\mathbf{K_2}$.** (Okay to find only the 2×2 nonzero part if you like, but you must clearly state where it goes in the full stiffness matrix K. It should be a check on your answer to part [i].)

iii Find the load vector F.

iii On the whole domain $[0, 1]$, the the exact solution is $u(x)$, and the finite element approximation is $U(x) = U_1\phi_1(x) + U_2\phi_2(x) + U_3\phi_3(x)$. True or False:

"For this special example, these functions are always equal, i.e. $U(x) = u(x)$".

c Now suppose the boundary conditions are changed to $u(0) = 0$, $\mathbf{u'(1) = 1}$. Create a new finite element equation $K\,U = F$ that models this new problem. Use the same h.

i What is the weak form?

ii Find the load vector F.

iii True or False:

"The matrix K in this new problem is the same as the matrix K from part (b)".

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– END OF QUIZ –